This is a digital copy of a book that was preserved for generations on library shelves before it was carefully scanned by Google as part of a project to make the world's books discoverable online.

It has survived long enough for the copyright to expire and the book to enter the public domain. A public domain book is one that was never subject to copyright or whose legal copyright term has expired. Whether a book is in the public domain may vary country to country. Public domain books are our gateways to the past, representing a wealth of history, culture and knowledge that's often difficult to discover.

Marks, notations and other marginalia present in the original volume will appear in this file - a reminder of this book's long journey from the publisher to a library and finally to you.

Usage guidelines

Google is proud to partner with libraries to digitize public domain materials and make them widely accessible. Public domain books belong to the public and we are merely their custodians. Nevertheless, this work is expensive, so in order to keep providing this resource, we have taken steps to prevent abuse by commercial parties, including placing technical restrictions on automated querying.

We also ask that you:

+ *Make non-commercial use of the files* We designed Google Book Search for use by individuals, and we request that you use these files for personal, non-commercial purposes.

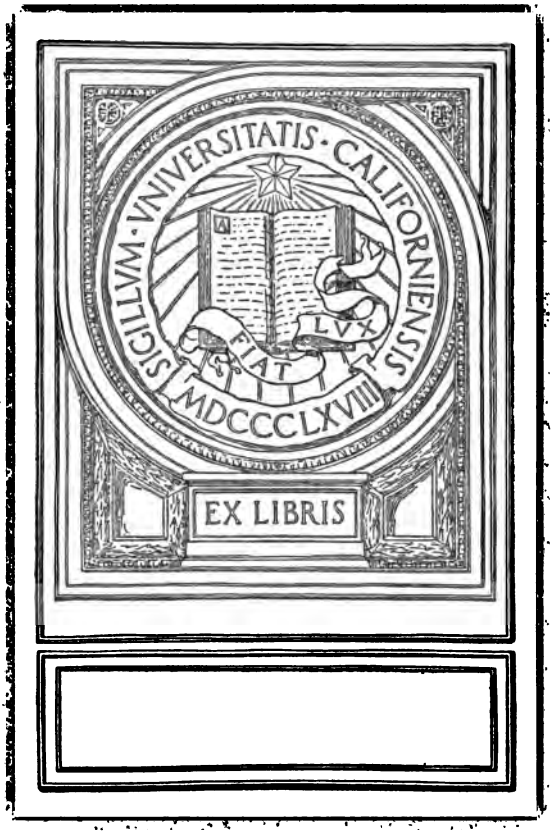
+ *Refrain from automated querying* Do not send automated queries of any sort to Google's system: If you are conducting research on machine translation, optical character recognition or other areas where access to a large amount of text is helpful, please contact us. We encourage the use of public domain materials for these purposes and may be able to help.

+ *Maintain attribution* The Google "watermark" you see on each file is essential for informing people about this project and helping them find additional materials through Google Book Search. Please do not remove it.

+ *Keep it legal* Whatever your use, remember that you are responsible for ensuring that what you are doing is legal. Do not assume that just because we believe a book is in the public domain for users in the United States, that the work is also in the public domain for users in other countries. Whether a book is still in copyright varies from country to country, and we can't offer guidance on whether any specific use of any specific book is allowed. Please do not assume that a book's appearance in Google Book Search means it can be used in any manner anywhere in the world. Copyright infringement liability can be quite severe.

About Google Book Search

Google's mission is to organize the world's information and to make it universally accessible and useful. Google Book Search helps readers discover the world's books while helping authors and publishers reach new audiences. You can search through the full text of this book on the web at `http://books.google.com/`


SIGILLVM · VNIVERSITATIS · CALIFORNIENSIS
LVX
FIAT
MDCCCLXVIII
EX LIBRIS

SCHOOL ALGEBRA

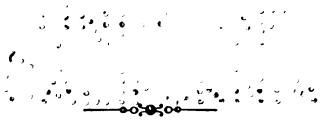
BY

JOHN M. COLAW, A.M.

AND

JOHN K. ELLWOOD, A.M.

AUTHORS OF A SERIES OF ARITHMETICS AND ALGEBRAS



RICHMOND

B. F. JOHNSON PUBLISHING CO.

1903

QA154
C57

COPYRIGHT, 1903,
BY J. M. COLAW AND J. K. ELLWOOD.

CAJORI

08-10-1e. F. Co.

PREFACE

In this book the arithmetical basis of algebra is made prominent. In the earlier stages the formulæ and results are frequently tested by arithmetical applications, and in passing to new ideas numerical illustrations are often used. In this way the transition from arithmetic to algebra is gradually and easily made.

The authors have endeavored to present the subject of algebra as simply and clearly as is consistent with strict mathematical form. This is especially noticeable in the definitions and in the proofs of principles. The tendency of pupils to use symbols in a purely mechanical way is obviated as far as possible by the clear and concise manner in which the matter in elementary operations is presented.

This book is well adapted to schools in which the study of algebra is begun. It is especially fitted for use in high schools, academies, and other preparatory schools. It will be found that the simpler subjects of algebra are given a broader treatment than is usual in books of this grade; but by omitting the proofs of principles and the latter half of the exercises and problems an easier course is secured. The complete course is sufficiently full to meet the requirements for entrance into any college or university.

The large number and variety of the exercises, and the abundance and practical character of the problems, many of

911276

which are drawn from the fields of science and everyday life, will commend themselves to the progressive teacher. The old-time "puzzles" are omitted, and the more difficult problems are placed in the latter part of the lists. Artificial difficulties, such as curious combinations of the symbols of aggregation, complicated fractions never met with in practice, elaborate expressions in factoring, unused combinations of indices, needlessly involved equations, etc., have been purposely omitted. By the elimination of these labor-wasting difficulties the student is given time for the important practical matter that is abundantly supplied.

The treatment of factoring and its application to the solution of equations is especially thorough. The difficulties which so frequently confuse the student as a result of following the axioms blindly are cleared away by a comprehensive treatment of Equivalent Equations. The Binomial and Remainder Theorems are presented in a simple manner at an early stage in order to give the student as quickly as possible the practical use of these important principles.

Many cautions on the errors that beginners are likely to fall into are given throughout the book, especially in the illustrative solutions. These solutions are intended to be models in neatness and orderly procedure, while the frequent use of "checks" will lead to accuracy and tend to make the student independent of book and teacher.

The value of early and frequent use of graphical representation to make algebraic abstractions seem real is recognized. Particular attention has therefore been given to the Graphs of Equations; and to the end that the graphical treatment may have practical value, it is presented where most needed to

throw light upon the algebraic solution. In explaining the conditions which render equations simultaneous, inconsistent, or impossible, the graphic method has been effectively employed.

The method of presentation is such as to lead the student into a clear understanding of the truths set forth, to think logically, to reason accurately, and to express his thoughts clearly.

The authors acknowledge their special obligation to Professor F. W. Duke, of Hollins Institute, Hollins, Va., for his able assistance in revising the various chapters of the book.

J. M. COLAW,
MONTEREY, VA.

J. K. ELLWOOD,
PITTSBURG, PA.

CONTENTS

	PAGE
INTRODUCTION	9
Literal or General Number	9
Symbols of Operation	10
The Equation	16
Axioms	19
Positive and Negative Numbers	28
ADDITION	36
Addition of Monomials	41
Addition of Polynomials	42
SUBTRACTION	45
Subtraction of Polynomials	51
Symbols of Aggregation	53
MULTIPLICATION	58
Monomials by Monomials	64
Polynomials by Monomials	64
Polynomials by Polynomials	66
Powers	71
Important Products	73
The Binomial Theorem	76
DIVISION	80
Division of a Polynomial by a Monomial	82
Division of a Polynomial by a Polynomial	83
Important Quotients	88
The Remainder Theorem	91
Equations and Problems	97
FACTORS	104
Solution of Equations by Factoring	120

PAGE

HIGHEST COMMON FACTOR 123

LOWEST COMMON MULTIPLE 130

FRACTIONS 134
Reduction of Fractions 136
Addition and Subtraction 143
Multiplication 147
Division 150
Complex Fractions 153

SIMPLE EQUATIONS WITH ONE UNKNOWN NUMBER 158
Integral Equations 162
Equations containing Fractions 164
Proofs of Important Theorems 169
Discussion of the Solutions of Problems 185

SIMPLE EQUATIONS WITH TWO OR MORE UNKNOWN NUMBERS . 188
Equivalent Systems of Equations 199
Linear Equations with Three or More Unknown Numbers . 210

INDETERMINATE EQUATIONS 217
Graphs of Linear Equations 221

INVOLUTION AND EVOLUTION 227
Involution 227
Evolution 230

THEORY OF EXPONENTS 244
Positive Integral Exponents 244
Negative and Fractional Exponents 245
Laws for any Rational Exponents 250

RADICALS AND SURDS 258
Reduction of Surds 259
Addition and Subtraction of Surds 262
Multiplication of Surds 264
Division of Surds 266
Radical Equations 270

PAGE

IMAGINARY AND COMPLEX NUMBERS 275
Addition and Subtraction of Imaginaries 279
Multiplication of Imaginaries 280
Division of Imaginaries 281

QUADRATIC EQUATIONS WITH ONE UNKNOWN NUMBER . . . 283
Incomplete or Pure Quadratics 284
Complete or Affected Quadratics 287
Equations in the Quadratic Form 297
Roots of the Quadratic Equation 303

SIMULTANEOUS QUADRATIC EQUATIONS 316
Simultaneous Quadratics with Three Unknowns . . . 332
Graphs of Quadratic Equations 338

INEQUALITIES 345

RATIO, PROPORTION, AND VARIATION 351
Ratio 351
Proportion 354
Variation 361

SERIES 368
Arithmetical Progression 369
Geometrical Progression 376
Harmonic Progression 385

LIMITS, INFINITY, INDETERMINATE FRACTIONS 388
Limits 389
Infinity 390
Indeterminate Fractions 392
Discussion of the Solutions of Problems 394

LOGARITHMS 399
Properties of Logarithms 409
Some Applications of Logarithms 413

THE BINOMIAL THEOREM 419
Proof for Positive Integral Exponents 419

SUPPLEMENTARY PROBLEMS 424

SCHOOL ALGEBRA

INTRODUCTION

1. In passing from arithmetic to algebra the meaning of number and the method of representing it are extended, but there is nothing contradictory to what has already been learned in arithmetic. Algebra, like arithmetic, treats of number, and may be regarded as generalized arithmetic.

The symbols, 1, 2, 3, etc., are retained in algebra with their arithmetical meanings, and the same symbols, $+$, $-$, $\times$, $\div$, ( ), $=$, are used in each. Fractions, powers, and roots have the same as well as an extended meaning, and are written in the same form.

LITERAL OR GENERAL NUMBER

2. An important difference between arithmetic and algebra comes from the frequent and extended use in the latter of *letters* to represent *numbers*. Just as, in interest problems, p may stand for principal, r for rate per cent, t for the time, etc., so in any case such symbols as a, b, x, y, may be used to represent *any numbers whatever*.

In arithmetic we speak of 6 feet, meaning a *certain number* of feet, or of \$10, meaning a *certain number* of dollars; in algebra we speak of n feet, meaning *any number* or an *unknown number* of feet, or of x dollars, meaning *any number* or an *unknown number* of dollars.

3. Numbers represented by letters are called **Literal** or **General Numbers.** The reasoning is the same whether numbers are represented by letters or by figures.

Thus, if a stands for the number of pupils in a room, then $2a$ stands for twice this number, $3a$ for three times this number, etc.

EXERCISES

4. 1. If n stands for the number of books in my library, what is the meaning of $3n$? Of $5n$? Of $6n$? Of $\frac{1}{2}n$?

2. If x, y, and z stand for the cost of a horse, a cow, and a sheep respectively, for what does $x + y$ stand? $x + y + z$? $2x + 3y + z$?

3. If $x = 5$, $y = 9$, and $z = 7$, what is the value of $x + y$? Of $x + y - z$? Of $7x - 5z$?

4. If in a number of two digits, the digit in the ones' place is 5, and the digit in the tens' place is 2, the number is $2 \times 10 + 5$. Write a number containing a ones and b tens. What is the number if $a = 5$ and $b = 3$?

If $a = 2$, $b = 3$, $c = 4$, $d = 5$, find the values of the following:

5. $2a - b + d$.
6. $4a + 3b - 2c - d$.
7. $7a - b + 5c - 6d$.
8. $10a + 10b - 10d$.
9. $(12a + 12b) \div 5d$.
10. $(8d - 4b) \div 2c$.
11. $20d - 9(a + b + c)$.
12. $15c - 10(b + a) - 2d$.
13. $(3a \times 2b) + c - 8d$.
14. $(5c \times 3a) - 20b - 10c$.
15. $(15b \times 2a) \div (3c \times 3d)$.
16. $(25b \div d) \times (15a \div 2b)$.

SYMBOLS OF OPERATION

5. As in arithmetic, the **Symbol of Addition,** $+$, called *plus*, is placed before the number to be added.

Thus, $a + b$, read *a plus b*, means that b is to be added to a, just as $4 + 3$ means that 3 is to be added to 4.

6. The **Symbol of Subtraction**, $-$, called *minus*, is placed before the number to be subtracted.

Thus, $a - b$, read *a minus b*, means that b is to be subtracted from a.

7. The **Symbol of Multiplication**, $\times$, read *multiplied by*, when placed between two numbers or expressions, indicates that the one on the left is to be multiplied by the one on the right. A dot ($\cdot$) is sometimes used instead of the sign $\times$, but more frequently multiplication is indicated by the absence of all symbols, especially where only letters or a numeral and one or more letters are used.

Thus, $a \times b$, $a \cdot b$, and ab all indicate the product of a multiplied by b. Twice the product of a by b is usually written $2\,ab$, and read *two ab*. In indicating the multiplication of numerals, however, we cannot omit the sign $\times$, as that would change the meaning of the expression. Thus, 4×7 cannot be written 47. Why not?

8. The **Symbol of Division**, $\div$, read *divided by*, when placed between two numbers, shows that the one on the left is to be divided by the one on the right. The colon and the fractional form are also used to indicate division.

Thus, $a \div b$, $a : b$, and $\frac{a}{b}$ all mean the quotient of a divided by b.

9. The expression $(a + b)\,k$, or $[a + b]\,k$, means that $a + b$ is to be multiplied by k. The *parentheses* and the *brackets* inclosing $a + b$ are called **Symbols of Aggregation**. The *bar* $\overline{}$ and *braces* $\{\ \}$ are also used.

$(4 + 3)\,a = 7\,a$; $(5 - 2)\,a = 3\,a$; $(4 + 2) - (8 - 5) = 6 - 3$.

10. Custom has established the following **Conventional Order** for performing the operations where several are involved, as in the expression $a + b \times c \div d - e$: First, the multiplications and divisions are to be performed *successively from left to right;* second, the additions and subtractions in the same order.

The order will be seen in the following:

(*a*) $5+2\times6\div3-4=5+12\div3-4=5+4-4=5.$

(*b*) $5+12\div2\times3-6=5+6\times3-6=5+18-6=17.$

(*c*) $5+12\div(2\times3)-6=5+12\div6-6=5+2-6=1.$

If parentheses appear in the expression, the indicated operations within the parentheses must first be performed, as in (*c*) above.

EXERCISES

11. 1. The sum of a and b is indicated thus: $a+b$. What is the sum of a and c? Of b and d?

2. One pail holds a quarts, another $3a$ quarts. They both hold —— quarts.

3. May has b yards of ribbon, Helen has a yards. How many yards have they both?

4. $3b$ years is $2d$ years less than my age. I am —— years of age.

5. A table is m feet long and n feet wide. What is its perimeter, or distance around the table?

6. Taking b from a is indicated thus: $a-b$. How can you express the taking of c from a? d from b?

7. John took a quarts of milk out of a pail containing b quarts. There remained —— quarts.

8. A pencil was m inches long. I used n inches of it. What does $m-n$ inches indicate?

9. Homer had $\$a$ and Frank had $\$b$. They spent $\$c$. What indicates the sum they had left?

10. The product of a by b is ab. What is the product of a by c? Of b by d? Of 2 by a?

11. At $\$c$ each, what will d sheep cost?

12. How many shoes will be needed for b boys if each boy gets 2 shoes?

13. A field is m rods long and n rods wide. It contains —— square rods.

14. A dog ran $8a$ rods and a rabbit ran $5b$ rods. If $a = 7$ and $b = 10$, the dog ran —— rods farther than the rabbit. If $a = 3$ and $b = 5$, which ran the farther?

15. If $m \div n$ or $\frac{m}{n}$ indicates the quotient of m divided by n, what is the quotient of a divided by b? Of $3x$ divided by 3? By x?

16. A man gave \$$b$ to c boys. If each boy got the same sum, what was that sum?

17. The area of a field is a square rods; its length is d rods. Its width is —— rods.

18. Complete the following:

$8 + 5 \times 2 - 12 =$ $\qquad$ $12 - 5 + 2 =$

$4a + 5b - 2a =$ $\qquad$ $12 - (5 + 2) =$

$3 + 2 \times 6 \div 3 - 5 =$ $\qquad$ $7 + 3(5 - 2) =$

$a + 6b \div 2 + 3a =$ $\qquad$ $16 + (4 + 2) \times 3 =$

$2b + 2(a + b) - b =$ $\qquad$ $18 + (10 + 6) \div 2 =$

$15a + 3(b + a) \div 3 =$ $\qquad$ $18 - (10 + 6) \div 2 =$

12. A collection of letters, or of letters and other number symbols, connected by one or more of the signs of operation ($+$, $-$, $\times$, $\div$, etc.) is called an **Algebraic Expression.**

Thus, $2x + a$ and $3abc$ are algebraic expressions. It should be observed that a means $a \times 1$, or one a.

13. The parts of an algebraic expression connected by the signs $+$ and $-$ (neither sign being within parentheses) are called its **Terms.** If the expression contains neither sign, it has but one term and is called a **Monomial.**

Thus, $4ab$, $\frac{1}{3}x$, and $5m$ are monomials. In the expression $2a + 3b - 5c$, the three terms are $2a$, $3b$, and $5c$, and each taken by itself is a monomial. In $ax + 2(a + b) - c$, there are three terms, $2(a + b)$ being treated as a single term.

14. An algebraic expression made up of several terms is called a **Polynomial,** meaning *many-termed.*

15. A polynomial of two terms is called a **Binomial,** and one of three terms a **Trinomial.**

Thus, $2a + 5d$ is a binomial ; $3m + 4n - 2a$, a trinomial.

16. In the expression abc, the a, the b, and the c are called the **Factors** of the expression. An expression is a **Multiple** of any one of its factors, or of the product of any two or more of them.

In the expression $2ac + 3bc$, point out the terms; also the factors. How are factors distinguished from terms?

17. In a product any factor, or product of factors, is called the **Coefficient** of the rest of the product. In practice, the word is usually applied only to some factor whose *numerical value* is known or expressed, and which appears first in the product. Cf. Art. 100.

Thus, in the expression $2abc$, 2 is the coefficient of abc, $2a$ of bc, etc. Since $a = 1a$, the coefficient 1 is understood before any letter when no coefficient is expressed.

18. The product of two or more equal factors is called a **Power.** The product of two equal factors is called the *second power;* of three equal factors, the *third power;* and so on.

19. The number symbol that shows how many times the equal factor enters into a power is called an **Exponent.**

Thus, a^2 is read "a square," or "a to the second power," and means $a \times a$; a^3 is read "a cube," or "a to the third power," and means $a \times a \times a$; a^4 is read "a to the fourth power" ; and so on.

How is the exponent distinguished from the coefficient? In the expression $3ab$, of what is 3 the coefficient? What is the exponent of b?

Since b may be regarded as taken once as a factor to make itself, b is regarded as meaning b^1. Any letter is considered as having the exponent 1 when no exponent is expressed.

20. We have learned in arithmetic that one of the equal factors of a number is called a **Root** of the number. The *square root* is one of the two equal factors; the *cube root* is one of the three equal factors; and so on.

In expressions like "the square root of 3" we shall see that the word *root* has a broader meaning than that given in the ordinary definition, since 3 has not *two equal* integral factors.

21. As in arithmetic, evolution is indicated by the symbol $\sqrt{}$, said to have had its origin in *r*, the initial letter of *radix* (Latin, *root*); but it will be found that both involution and evolution are more frequently indicated by exponents.

EXERCISES

22. 1. Write the expression which means that x is to be multiplied by the sum of 5 and 2.

2. Indicate the multiplication of y by the difference between 7 and 4.

3. Is the expression $5ab + 10bc - 5bx$ a polynomial? What other name may be given to it? What factors are found in each term?

4. Write an expression having four terms and the factor ax.

5. Write a trinomial containing the factor $3b$.

6. Name the terms in the expression $6x - 2(a - b) + cd$.

7. Write a monomial whose factors are 2, a, x, and 3.

8. Write an expression which is a multiple of 5, b, c, 2, and y. Is it a monomial? Why?

9. Write an algebraic expression containing four signs of operation. How many terms does it contain?

10. Why is the expression, $x - 3(a + b)$, not a trinomial?

11. Write a monomial containing the factor a, the coefficient 5, and the exponent 2.

12. Write a binomial, each term containing the exponent 3 and the coefficient 4.

13. Use symbols to express the following:

Four a square plus three times b square.

Five a cube plus twice the third power of b.

Ten x cube minus six times the sum of b and c.

Six a times y cube plus four times the product of b and y.

14. What is the coefficient of x in $\frac{x^2}{3}$? The exponent?

15. What is the meaning of the expression abc? Of 125? What is the value of abc, if $a = 1$, $b = 2$, $c = 5$?

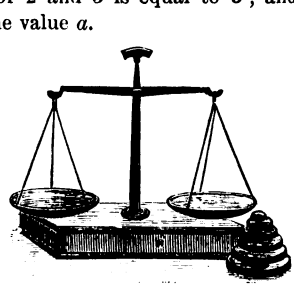
16. What is the value of each term in the expression, $2a \times b \div d \times c - a$, if $a = 1$, $b = 2$, $c = 3$, $d = 4$? What is the value of the polynomial?

THE EQUATION

23. The **Symbol of Equality**, $=$, is read *is equal to*, and indicates that the two numbers or expressions between which it is placed are equal in value.

Thus, $2 + 3 = 5$ means that the sum of 2 and 3 is equal to 5; and $x = a$ means that x is equal to a, or has the value a.

24. 1. When will a scalebeam balance?



2. If a weight be added to one pan, what must be added to the other to preserve the balance?

3. If a weight be taken from one pan, what must be done to preserve the balance?

4. What does the expression $2x + 3 = 11$ mean?

5. If we add 1 to the first side of the equality, how much must be added to the second side to make the sides equal again?

6. If we take 3 from the first side, what must be done to the second to preserve the equality?

7. Is there any difference in value between $7 + 3$ and $15 - 5$? Then $7 + 3 = 15 - 5$.

8. How much is 6 times $(7 + 3)$? How much is 6 times $(15 - 5)$? Are the products equal?

9. Then what may be done to both sides of an equality without destroying the equality?

10. What is the quotient of $(7 + 3) \div 2$? Of $(15 - 5) \div 2$? Are the quotients equal?

11. Then what else may be done to both sides without destroying the equality?

12. A 10-acre field is worth \$100 an acre, and a 20-acre field is worth \$ 50 an acre. In what respect are the two fields equal?

25. From the preceding exercises it will be seen that the two sides of any equality are like the weights in the two pans of a balanced scalebeam, and that, in general, any change made in one side requires a like change in the other if the equality is to be preserved.

26. When any two numbers or expressions are connected by the symbol of equality, the whole expression thus formed is broadly called an equation. This definition includes both the identity and the equation proper.

27. In the stricter sense in which the word is used in algebra, an **Equation** is an equality that exists only for particular values of the letters representing the *unknown numbers*. These values are called the *roots* of the equation.

Thus, $3x + 2 = 8$ is an equation because the equality exists only when x has the value 2. If $x = 3$, there is no equality.

28. An **Identity** is an equality that exists for any value that may be substituted for the letters involved.

Thus, $9a + 3a = (3 \times 4)a$ is an identity because the equality exists for *any* value that a may have. Any equality that involves numerals only is an identity, as $2 + 3 = 5$.

1. An equation states a condition, and the values of the unknown numbers are to be found. An identity states that one of two expressions is equal to the other, or can be reduced to the other; it is to be proved. When the roots of an equation are substituted for the letters representing its unknown numbers, the equation reduces to an identity.

2. An identity is indicated by the symbol $\equiv$; as, $a(a-b) \equiv a^2 - ab$. It is used in this book in all cases where it is desired to emphasize the fact that any particular equality is an identity.

29. The part of an equation that is written *before* the sign $=$ is called the *first member*, and that written *after* the sign, the *second member*.

Thus, in the equation $x + a = 10$, $x + a$ is the first member. What is the second member?

The two members may differ widely in *form*, but in *value* they must be the same.

30. Finding the values of the unknown numbers — the roots of an equation — is called *solving the equation*. If the value found for the unknown number is substituted in the original equation, and the equation reduces to an identity, the value of this unknown number (called the *root of the equation*) is said to be *verified*. This substitution must always be made in the original equation.

31. The extensive use of the equation is the most characteristic feature of algebra; it makes possible the solution of practical problems otherwise insolvable.

32. The **Symbols of Inequality** are $>$, meaning *is greater than*, and $<$, meaning *is less than*. The symbols $\neq$, *not equal to*; $\not<$, *not greater than*; and $\not>$, *not less than*, also, are used in algebra.

Thus, $a > b$ is read "a is greater than b"; $c < d$ is read "c is less than d"; etc.

33. A **Proposition** is a statement of a truth to be demonstrated or of something to be done. Propositions are either *theorems* or *problems*.

34. A **Theorem** is a statement of a truth to be demonstrated; as, show that $a^m \times a^n \equiv a^{m+n}$.

35. A **Problem** is a statement of something to be done; as, required the product of $(x + y)$ by $(x - y)$.

AXIOMS

36. General statements which are so obvious that their truth may be taken for granted are called **Axioms**.

The following are frequently met with in algebra:

1. **Expressions which are equal to the same thing or to equal things are equal to each other.**

For example, if $x + 2 = 5$ and $8 - x = 5$, then $x + 2 = 8 - x$.

2. **If equals are added to equals, the sums are equal.**

For example, if $a = b$, then $a + c = b + c$.

3. **If equals are subtracted from equals, the remainders are equal.**

For example, if $x + 3 = 5$, $x = 5 - 3 = 2$.

4. **If equals are added to unequals, the sums are unequal.**

For example, if x is greater than 3, $x + 2$ is greater than $3 + 2$, the first being greater both before and after the addition of the equals.

5. **If equals are subtracted from unequals, the remainders are unequal.**

For example, if x is less than 9, $x - 2$ is less than $9 - 2$.

6. **If equals are multiplied by equal numbers, the products are equal.**

For example, if $x = 3$, $2x = 3 \times 2 = 6$.

7. **If equals are divided by equal numbers** (not zero), **the quotients are equal.**

For example, if $4x = 8$, $2x = 8 \div 2 = 4$.

8. **Like powers of equal numbers are equal.**

For example, if $x = 3$, $x^2 = 9$.

9. **Like roots of equal numbers are** (arithmetically) **equal.**

For example, if $x^2 = 25$, $x = 5$. The algebraic sense in which they are not equal will be seen in a later discussion.

37. Principles and relations are expressed in **Algebraic Language** by means of the signs and symbols of algebra. Some of the advantages of such symbolic statements are seen in the following:

1. In arithmetic, the verbal statement that twice a certain number, added to 10, equals 150 is more briefly expressed as follows:

$$2 \times \text{some number} + 10 = 150.$$

In algebra we express it more conveniently, thus:

$$2x + 10 = 150.$$

2. The following is a principle of subtraction: *The sum of the subtrahend and the remainder is equal to the minuend.* Now, if we represent the subtrahend by s, the remainder by r, and the minuend by m, we may express this principle in algebraic language as follows: $s + r = m$.

QUERIES. 1. Does algebraic language express all that verbal statements can express?

2. Does it do so much more briefly?

3. Are the symbolic statements an aid to the understanding? Why?

38. In the solution of problems it is frequently necessary to change terms of the equation from one side to the other in order to simplify it. The operation is in reality addition or subtraction, involving an application of axioms 2 and 3, Art. 36.

1. In the equality $\$16 - \$5 = \$8 + \$3$, transpose $\$5$ to the second member.

Adding $\$5$ to each member (axiom 2), we have $\$16 - \$5 + \$5 = \$8 + \$3 + \$5$.

But since $\$16 - \$5 + \$5 = \$16$, we may write the equality $\$16 = \$8 + \$3 + \$5$.

Compare this equality with the one given and note the changes.

It will be noticed that $(-\$5)$ has disappeared from the first member, while $(+\$5)$ appears in the second.

2. In the equation $5x = 3x + 8$, if we subtract $3x$ from each member (axiom 3), we have

$$5x - 3x = 3x - 3x + 8.$$

Since $3x - 3x = 0$, we may write the equation

$$5x - 3x = 8.$$

It will be noticed that $3x$ has disappeared from the second member, and appeared in the first, with a different sign before it.

39. Any term may be transposed from one member of an equation to the other by dropping it from the member in which it stands, and writing it in the other, with a different sign.

Transpose so that only odd numbers will be in the first member:

1. $7 - 4 = 8 - 5$.
2. $2 + 9 = 1 + 10$.
3. $19 + 6 = 30 - 5$.
4. $5 \times 3 - 2 \times 5 = 1 + 4$.
5. $39 \div 3 - 4 \div 2 = 22 \div 2$.
6. $7 + 4 - 8 = 26 - 25 + 2$.

Transpose so that only like terms will be in each member:

7. 3 times a number $- 5 =$ 2 times the number.
8. 5 times A's money $- \$10 =$ 3 times A's money $+ \$20$.
9. $\frac{1}{4}$ of my money $= \$9 - \frac{1}{5}$ of my money.
10. $\frac{3}{5}$ of A's age $+ 4$ years $= \frac{1}{4}$ of A's age $+ 11$ years.
11. 3 rods $+ 2$ yards $= 2$ rods $+ 7\frac{1}{2}$ yards.

Transpose all terms containing x to the first member, and all others to the second member:

12. $7x - 5 = 3x + 15.$
13. $5 + 6x - 9 = 2x + 8.$
14. $5x + 6 - 8 = 4 + 3x.$
15. $9x - 36 = 24 - 6x.$
16. $8x - 2y = 14y - 16x.$
17. $3y + 10x = 10 + 11x.$

40. After like terms have been collected into one member, the equation may often be made still more simple by performing the operations indicated.

Thus, the equation $5x - 3x = 24 + 6$ may be written $2x = 30.$

Perform indicated operations:

1. 4 times a number − 3 times the number = 10 + 5.
2. 2 × B's money + 3 times B's money = \$ 100 − \$ 20.
3. $7 \times 5 - 64 \div 16 - 11 = 5 \times 3 + 5.$
4. $6 + 5 \times 6 + 18 - 2 = 2 + 25 \times 2.$

This may be called *uniting like terms.*

Supply the wanting term:

5. $32 - 5 \times 3 + 45 \div 9 = (\quad).$
6. $46 - 18 + 16 + (\quad) = 30 + 7 \times 10.$
7. $75 + 13 \times (\quad) - 51 = 64 - 2 \times 7.$
8. $\frac{3}{4} + \frac{1}{2} \times \frac{2}{3} - (\quad) = 2 - 2 \times \frac{1}{2}.$

PROBLEMS

41. Expressing the conditions of a problem in the form of an equation is called *stating the problem.* The several steps of the *analysis* may be expressed in a series of equations, each derived from the one preceding, by certain changes according to principles.

42. The **Signs of Deduction** most used are $\therefore$, read *therefore,* and $\because$, read *since.*

1. What number increased by $\frac{1}{2}$ of itself equals 18?

(*a*) What is to be found? (A certain number.)

(*b*) Then what shall x represent? (The required number.)

(*c*) For what number can two expressions be found? (In the problem, 18 and "the number + $\frac{1}{2}$ of the number" are expressions for the same number.)

(*d*) How do we state the equality of the two expressions in algebraic language? ($x + \frac{1}{2}x = 18$.)

2. Five times a given number is equal to twice the number increased by 36. Find the number.

FIRST SOLUTION

Stated in the briefer arithmetical form, we have

five times the number = twice the number + 36.

Subtracting "2 × the number" from these equals, we have

5 × the number − 2 × the number = 36.

Combining (uniting terms), we have

3 × the number = 36.

Dividing these equals by 3, we find that

the number = 12.

SECOND SOLUTION

Let x = the number.

Then $5x = 5$ times the number,

and $2x = 2$ times the number.

Hence, $5x = 2x + 36$.

Subtracting $2x$ from each member (axiom 3),

$5x - 2x = 36$.

Uniting terms, $3x = 36$.

Dividing by 3 (axiom 7), $x = 12$.

CHECK. Substituting 12 for x in the first equation, we obtain the identity,

$60 = 24 + 36$.

In the solution of any problem, the *number to be found* is the one that must be represented by a letter. This letter may be treated just as the number itself would be if known.

3. The sum of two numbers is 36. One number is 5 times the other. Find the smaller number.

Let x = the smaller number.
Then $5x$ = the larger number.
$\therefore$ $x + 5x = 36$, the sum. (Why ?)
Uniting terms, $6x = 36$.
Dividing by 6 (axiom 7), $x = 6$.

CHECK. Substituting 6 for x in the first equation, we obtain the identity, $6 + 30 = 36$.

4. A coat and a hat cost \$36. The coat cost 5 times as much as the hat. What was the cost of the hat?

Let x = the *number* of dollars the hat cost.
Then $5x$ = the number of dollars the coat cost.
$\therefore$ $x + 5x$ = the number of dollars both cost.
$\therefore$ $6x = 36$, (Why ?)
$x = 6$. (Why ?)
$\therefore$ the hat cost \$6.

It should be observed that when it is found that $x = 6$, it is known that the result is \$6, since x stands for the *number* of dollars.

5. One third of my sheep equals one ninth of them plus 8. How many have I?

Let x = the total number of sheep.
Then $\frac{1}{3}x = \frac{1}{9}x + 8$ or $\frac{x}{3} = \frac{x}{9} + 8$.

$\frac{1}{3}x - \frac{1}{9}x = 8$ or $\frac{x}{3} - \frac{x}{9} = 8$.

$\frac{2}{9}x = 8$ or $\frac{2x}{9} = 8$. (Why ?)

$\frac{1}{9}x = 4$ or $\frac{x}{9} = 4$. (Why ?)

$x = 36$ or $x = 36$. (Why ?)

$\therefore$ I have 36 sheep.

CHECK. Have the pupil give check.

6. What sum at 6% will amount to \$159 in a year?

Let x = the number of dollars in principal.

Then $.06\,x$ = the number of dollars in interest.

$\therefore\quad x + .06\,x = 159.$ (Why?)

$1.06\,x = 159.$

$x = 150.$ (Why?)

$\therefore$ the sum is \$150.

CHECK. $150 + 150 \times .06 = 159.$

7. A and B together have \$80, and A has 3 times as many dollars as B has. How much has each?

8. Five times a certain number is 24 more than 3 times the number. Find the number.

9. What number is that whose $\frac{2}{3}$ is 3 more than its half?

10. A farmer has 100 hens and chicks. Each hen has 9 chicks. How many of each has he?

11. Mr. E has gold dollars and silver dollars to the amount of \$30. He has one half as many silver dollars as gold dollars. How many gold dollars has he?

12. A and B together sold 728 bushels of corn, and B sold 3 times as much as A. How much did B sell?

13. Find a number such that if it be added to one third of itself the sum will be 60.

14. A city gained 10% in population in five years, and then had 27,500 inhabitants. What was the population before the gain?

15. Dick and Jack together have \$40, and Jack has \$10 more than Dick. How much has each?

16. The difference between two numbers is 48, and the greater number is 5 times the less. Find the smaller number.

17. By selling a watch for \$36 I gained $\frac{4}{5}$ of its cost. What was its cost?

18. A got $\frac{1}{4}$ of his father's fortune, and B got $\frac{1}{5}$ of it. If A got \$2000 more than B, what was the fortune?

19. If 18 is added to 6 times a certain number, the sum will be 22 less than $\frac{2}{3}$ of 150. Find the number.

20. Out of a total of 1452 votes, B received 3 times as many votes as C, and A twice as many as C. How many votes did each receive?

21. A and B are partners in business, and A's capital is \$500 less than twice B's capital. Their total capital is \$11,000. How much has each?

22. Marvin solved a certain number of problems, and Owen solved 12 less than twice as many. Both solved 72. How many did each solve?

23. A has \$4 less than 3 times what B has, and both together have \$124. How much has each?

24. Three boys, A, B, and C, have 77 marbles; B has 10 more than A, and A has 8 more than C. How many has each boy?

25. If, with the money I now have, I had 3 times as much, and \$25 more, I should have \$125. How much money have I?

26. Divide \$1000 between B and C so that B may have $\frac{2}{3}$ as much as C.

27. Twice a certain number is equal to the number increased by $\frac{1}{2}$ of itself, $\frac{1}{3}$ of itself, and 18 more. Find the number.

28. Three fourths of A's money diminished by \$20 is equal to $\frac{1}{3}$ of his money increased by \$5. Find A's money.

29. The sum of two numbers is 50, and the difference is 20. What are the numbers?

Let x = the smaller number.

30. Tom, Ned, and Frank together caught 36 fish. Tom caught twice as many as Ned, and Frank caught as many as both Tom and Ned. How many fish were caught by each?

31. A man doubled his capital each year for three years, and then had \$2800. What was his capital at first?

32. A rectangular field is 3 times as long as it is broad, and the perimeter is 128 rods. Find the length and breadth of the field.

33. A trader having increased his capital by 6% found that it amounted to \$4240. How much was it at first?

34. Find a number such that the sum of its half and its third may exceed the sum of its fourth and its fifth by 23.

35. In a certain theater there were 850 persons. There were 3 times as many men as women, and 4 times as many women as children. How many were there of each?

36. The sum of three consecutive numbers is 36. Find the numbers.

37. Find two numbers such that the larger shall be 4 times the smaller, and that 6 times the smaller shall equal 60 plus the larger.

38. Three leaden balls and two round marbles, all of same size, weigh 42 ounces. Find the weight of each, if a leaden ball weighs as much as four marbles.

39. Divide 54 into two parts such that one shall be to the other as 4 to 5.

40. A's farm is 3 times as large as B's, and both together lack just 10 acres of making a quarter-section, 160 acres. How many acres in each farm?

41. A school gives \$140 for three prizes. The first is double the second, and the second is double the third. What is the sum in each prize?

42. A farmer pays just as much for 2 horses as he does for 3 cows. If a cow costs \$15 less than a horse, what is the cost of each?

Let x = the number of dollars a cow costs.

43. A had 3 times as much money as B; after giving \$3 to B he had only twice as much. What had each at first?

44. Twice a certain number increased by 24 exceeds 80 by as much as the number itself is less than 100. Find the number.

45. A man left $\frac{1}{2}$ of his estate to his wife, $\frac{1}{10}$ for charity, $\frac{3}{8}$ to his children, and \$1400 to his servants. What was the amount of the estate?

46. A pole 36 feet long was broken into two unequal pieces, $\frac{3}{5}$ of the longer piece being equal to $\frac{3}{4}$ of the shorter. How long was each piece?

47. In riding a distance of 36 miles in 3 hours, a bicyclist goes a certain distance the first hour, $\frac{3}{4}$ as far the second hour, and $\frac{1}{2}$ as far the third hour. How far did he go the first hour?

POSITIVE AND NEGATIVE NUMBERS

43. Sometimes quantities that are measured by the same unit are of *opposite qualities*. Thus, assets and liabilities are both measured by the unit *dollar*, the readings of a thermometer above and below zero are given in *degrees*, and dates A.D. and dates B.C. are both given in *years*.

In the case of assets and liabilities, the unit *dollar* may be taken either as a dollar of assets or as a dollar of liabilities; if as a dollar of assets, then assets are regarded as *positive;* and liabilities, for the sake of distinction, as *negative*. In order to represent quantities that have opposite qualities, we need to extend the idea of number as given in arithmetic so as to include numbers that count *negative units*.

44. The numbers arithmetic deals with are *greater than zero*, and are called **Positive Numbers**; but algebra treats also of numbers that in relation to positive numbers are regarded as *less than zero*, and these are known as **Negative Numbers**.

45. The primary notion of a negative number is that of one which, when taken with a positive number of the same kind, goes to diminish it, cancel it, or reverse it.

Thus, liabilities neutralize assets, thereby diminishing the net assets, canceling them, or leaving a net liability.

46. Negative numbers may be regarded as arising through the extension of the operation of subtraction to the case in which the minuend is less than the subtrahend, which from an arithmetical point of view, is impossible. Note the following:

$6 - 4 = 2.$

$5 - 4 = 1.$

$4 - 4 = 0.$

$3 - 4 = -1$; that is, a number one unit less than 0.

$2 - 4 = -2$; that is, a number two units less than 0.

Observe that as the minuend decreases by 1, 2, or more units, the subtrahend remaining the same, the remainder decreases by an equal number of units, becoming 0 when the minuend is equal to the subtrahend. If, then, the minuend becomes less than the subtrahend by 1, 2, or more units, the remainder must decrease by an equal number of units, and therefore become less than 0 by 1, 2, or more units. But the operation of subtracting a greater number from a less is *possible only when numbers less than zero are introduced.*

47. The negative remainder, -1, does not mean that more units were taken from the minuend 3 than it contained; it merely shows that the subtrahend is 1 unit *greater than the minuend.*

48. The **Absolute Value** of a number is the number of units contained in it without regard to their *quality.* Since letters are used to represent numbers which may have any values whatever, they can represent either positive or negative numbers.

49. In the expression $6 - 4$, the *minus sign* indicates that 4 is to be taken from 6. It is a symbol of *operation*, and does *not* show 4 to be a negative number. Both 6 and 4 are positive numbers.

But the same sign has another use, namely, to denote negative numbers; and the sign $+$ is used to denote positive numbers. When so used these signs are symbols of *quality*, and do *not* indicate any operation whatever.

Thus, -4 means a number four units less than 0, and $+4$ a number four units greater than 0.

50. Positive and negative numbers are called *opposite* numbers, and may represent any quantities that are opposite in their relation to each other.

Thus, degrees above zero on a thermometer may be called positive; degrees below zero, negative; distance east, positive; distance west, negative; assets, positive; liabilities, negative.

51. Cash received and cash spent are opposite quantities, and may be represented by positive and negative numbers.

Thus, \$ 50 received may be represented by $+\$ 50$,

and \$ 50 spent may be represented by $-\$ 50$.

Monday, received,	\$ 25, or	$+\$ 25$
Monday, spent,	8, or	-8
Tuesday, spent,	13, or	-13
Wednesday, received,	9, or	$+9$
Wednesday, spent,	16, or	-16
Thursday, spent,	7, or	-7
Thursday, received,	10, or	$+10$
Friday, cash on hand,	\$ 0	\$ 0

A man's cash account might be kept as in the left-hand column, or as in the right. Friday he looks over his figures to see how much cash he ought to have on hand. He adds the sums received, which amount to \$ 44, and the sums expended, which amount to \$ 44. These cancel each other. That is, \$ 44 received united with \$ 44 spent is equal to neither cash on hand nor debt; or

\$ 44 received + \$ 44 spent = 0.

Or, he may add the positive numbers in the column to the right, getting $+\$ 44$, and the negative numbers, getting $-\$ 44$. This may be expressed algebraically thus:

$$+\$ 44 + -\$ 44 = 0.$$

52. We have already seen that zero is the *difference* between two equal numbers. From the preceding article we learn that it is also the *sum* of two equal and opposite numbers.

Thus, *gain* \$ 5 + *loss* \$ 5 = 0.

Or, in algebraic language,

$$+\$5 + -\$5 = 0.$$

Carefully examine the following statements:

20 dollars *gain* + 20 dollars *loss* = 0.

$$+20 + -20 = 0.$$

This result means neither *gain* nor *loss*.

20 dollars *gain* + 15 dollars *loss* = 5 dollars *gain*.

$$+20 + -15 = +5.$$

This result makes a net *gain* of 5 dollars.

20 dollars *gain* + 30 dollars *loss* = 10 dollars *loss*.

$$+20 + -30 = -10.$$

This result means a net *loss* of 10 dollars.

5 miles *east* + 5 miles *west* = the starting point.

$$+5 + -5 = 0.$$

This result means that the traveler has returned to the point from which he started.

10 miles *north* + 15 miles *south* = 5 miles south.

$$+10 + -15 = -5.$$

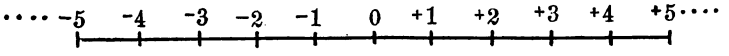
This result means that the traveler stopped 5 miles south of his starting point.

Make similar statements for each of the following:

1. \$ 80 gain, \$ 50 loss.
2. \$ 75 gain, \$ 100 loss.
3. 40 miles east, 30 miles west.
4. A rise of 20° in temperature, then a fall of 18°.
5. An army balloon ascended 3000 feet, then fell 1800 feet.

53. Zero is **considered** as having no sign, or as having both the plus and minus signs. It is the number from which positive and negative numbers are counted.

54. It thus appears that positive numbers may be **represented as standing on one side of zero, and negative numbers on the other.** Thus, if the divisions of the line below be taken as units of length, with zero at the midpoint, the positive (arithmetical) numbers will express the *successive repetitions of the unit to the right*, while the negative numbers, representing *equal measures*, but *opposite directions*, will as clearly express the *repetitions of the unit to the left.*



55. Although negative numbers arise as the result of enlarging our conception of subtraction, it appears from the foregoing that it is necessary to treat them apart from this particular operation. They not only serve to count *negative units*, as shown above, but also are demanded to make the principles of our number system general.

Negative results may or may not admit of a *concrete* interpretation in a particular problem; their *abstract* signification, however, is the same. Like fractional results, they may denote an impossibility by reason of the physical conditions imposed by the problem. For instance, we find no difficulty in dividing a 10-foot line into three equal parts, but we cannot separate 10 boys into three equal groups. However, the algebraic statement is the same in either case — to find x such that $3x = 10$. In like manner, the formal solution is the same in problems involving negative results, the possibility of a concrete interpretation depending upon the character of the particular problem.

NOTE. The development of the number idea has been traced through many centuries. First came the natural numbers, 1, 2, 3, 4, etc.; later the fraction; then the idea was extended to include the expressions $\sqrt{2}$, $\sqrt{3}$, $\sqrt[3]{5}$, etc., and much later the necessity of introducing negative numbers was recognized in order to solve equations like $x + a = 0$, or $x + 5 = 0$. Without enlarging our concept of number, such equations would have no meaning.

56. Positive and negative numbers, together with zero, are called **Algebraic Numbers.** They are as follows:

$$\cdots -5, -4, -3, -2, -1, 0, +1, +2, +3, +4, +5, \cdots.$$

The *sign of continuation*, $\cdots$, used here, is read "and so on"; it indicates that the series continues without end. When a number is written after the sign, the series continues only to that number.

Thus, 2, 4, 6, 8, $\cdots$ 40, is read "two, four, six, eight, and so on to forty."

57. Since any two such expressions as $+3$ and -3 represent opposite directions or qualities, but equal measures, they are said to have the same *absolute value.* From an inspection of a thermometer it is seen that the difference between $+3°$ and $-3°$ is $6°$; hence it is seen that we find the difference between a positive and a negative number, having the same absolute value, by *adding the absolute values.* The difference is found in the same way when the absolute values are different.

58. If to any number to the left of 0, as written in Art. 54, we add $+1$, we obtain the next right-hand number. Thus, $-3 + +1 = -2$; $-2 + +1 = -1$, and so on. Hence, if a number is increased by adding $+1$ to it, the numbers in Art. 54 increase from left to right.

That is,

$$-3 < -2,\ -2 < -1,\ -1 < 0,\ 0 < +1,\ +1 < +2, \text{ etc.}$$

We have, therefore, the following relations of positive and negative numbers:

I. Any positive number is greater than zero; while any negative number (as related to positive numbers) is less than zero.

II. Of two positive numbers the greater has the greater absolute value; while of two negative numbers the greater has the less absolute value.

EXERCISES

59. Read the following:

1. $-2 < +2$.
2. $-3 < 0$.
3. $+3 > 0$.
4. $-4 < -1$.
5. $-7 > -10$.
6. $0 < +2$.
7. $0 > -2$.
8. $+6 > +4$.
9. $+4 < +6$.
10. $-100 < +1$.
11. $+1 > -40$.
12. $+a > 0$.
13. $-a < 0$.
14. $-5b < -2b$.
15. $0 > -10a$.

Express in algebraic language:

16. $+5$ is greater than 0.
17. -5 is less than 0.
18. $+7$ is greater than $+3$.
19. -7 is less than $+3$.
20. 0 is greater than -6.
21. $+1$ is greater than $(3-7)$.
22. $+b$ is greater than 0.
23. $-b$ is less than 0.
24. $-4a$ is less than $-2a$.
25. $-6a$ is greater than $-12a$.

State the difference between:

26. $+4$ and -4.
27. $+5$ and -7.
28. $+8$ and $+6$.
29. -8 and -6.
30. -7 and 0.
31. -3 and $+1$.
32. $+2$ and 0.
33. -1 and $+5$.
34. -1 and 0.
35. 0 and $+4$.
36. 0 and $+2a$.
37. 0 and $-3b$.
38. $-5a$ and $+5a$.
39. $-6a$ and $-3a$.
40. $-3b$ and $+7b$.

41. What is the absolute value of $+7$? Of $+3$? Of -8?
42. What is the absolute value of x when $x = -6$?
43. What is the value of x when $x - +4 = 0$?
44. What value has x when $x - -4 = 0$?
45. What is the value of x when $x - +a = 0$?
46. When $12 - x = 0$, what is the absolute value of x?
47. What is the absolute value of x in $18 - 2x = 0$?
48. What is the value of $+a - +7$, when $a = 10$? When $a = 7$? When $a = 3$?
49. What is the value of $a - +5$, when $a = +9$? When $a = +5$? When $a = -5$?

50. Show that if $a = 7$ and $b = 5$, $(a + b)(a - b) \equiv a^2 - b^2$.

51. The reading of a thermometer was $^{+}20°$ and its mercury falls 30°. What is the reading now?

52. If I say a man is worth $^{-}\$1000$, what does the expression mean to you?

53. Indicate by symbols the difference between 100 years B.C. and 100 years A.D.

54. What is the combined weight, under water, of a piece of metal weighing 5 lb. and a piece of cork weighing $^{-}1$ lb., that is, pulls up with a force of 1 lb.?

Express in algebraic language:

55. The sum of any two numbers diminished by either one of them is equal to the other.

56. The sum of any three numbers diminished by any one of them is equal to the sum of the other two.

57. The minuend diminished by the remainder is equal to the subtrahend.

58. The product divided by the multiplier is equal to the multiplicand.

59. The area of a rectangle is equal to the product of its base by its altitude.

60. The area of a circle is equal to the square of its radius multiplied by π.

61. The square on the hypotenuse of a right-angled triangle is equal to the sum of the squares on the other two sides.

62. If s is the side and a the area of a square, what principle is expressed by $a = s^2$?

63. If b is the base, s the area, and a the altitude of a triangle, what principle is expressed by $s = \frac{1}{2}(a \times b)$?

64. If z is the hypotenuse, x the base, and y the altitude of a right-angled triangle, what principle is expressed thus, $z^2 - x^2 = y^2$?

ADDITION

60. 1. If a man is worth \$300 and earns \$100 more, what is he then worth? What is the sum of $^{+}\$300$ and $^{+}\$100$? $^{+}\$300 + {}^{+}\$100 =$

2. If a man begins the year \$300 in debt, and during the year incurs a debt of \$100, how much does he then owe? What is the sum of $^{-}\$300$ and $^{-}\$100$? $^{-}\$300 + {}^{-}\$100 =$

3. If a person has \$300 and owes \$100, what is his net capital? What is the sum of $^{+}\$300$ and $^{-}\$100$? $^{+}\$300 + {}^{-}\$100 =$

4. If I have \$500 and incur a debt of \$1000, what is my net capital? What is the sum of $^{+}\$500$ and $^{-}\$1000$? $^{+}\$500 + {}^{-}\$1000 =$

5. To find the sum in the last example, do we add or subtract the absolute values? In the first example? In the second? In the third?

6. Which of the four sums are positive? Which negative?

7. In which of the examples do the addends have like signs? Unlike signs?

61. **Addition** is the process of uniting one algebraic number with another into one aggregate. In this process the one number is said to be *added* to the other; the expression $a + b$ means that b is to be added to a.

In algebra the word *sum* is extended to include the result of adding negative numbers, as well as that of adding positive and negative numbers.

62. Numbers with Like Signs.

In algebra the process of adding two or more positive numbers, or two or more negative numbers, is the same as that of adding in arithmetic, except that the *sign of quality* is to be prefixed to the sum.

Thus, the sum of 2 positive units and 3 positive units is $(2 + 3)$ positive units, or 5 positive units; that is, $^{+}2 + {}^{+}3 = {}^{+}(2 + 3)$, or $^{+}5$.

In like manner, $^{-}2 + {}^{-}3 = {}^{-}(2 + 3)$, or $^{-}5$; that is, to add $^{-}2$ and $^{-}3$ means to add 2 and 3 and to prefix the sign $-$ to the sum.

Add the following:

1. $^{+}9$ to $^{+}16$.
2. $^{+}8$ to $^{+}27$.
3. $^{+}5$ to $^{+}45$.
4. $^{-}12$ to $^{-}8$.
5. $^{-}21$ to $^{-}4$.
6. $^{-}33$ to $^{-}7$.
7. $^{+}36$ to $^{+}64$.
8. $^{-}45$ to $^{-}30$.
9. $^{+}6\frac{1}{2}$ to $^{+}3\frac{1}{2}$.

63. Numbers with Unlike Signs.

The following equalities were considered in Art. 52:

$$^{+}20 + {}^{-}20 = 0. \quad (1)$$

$$^{+}20 + {}^{-}15 = {}^{+}5. \quad (2)$$

$$^{+}20 + {}^{-}30 = {}^{-}10. \quad (3)$$

In (1) we see that $^{+}20$ and $^{-}20$ cancel each other, that is, their *sum* is 0.

In (2) we see that 15 negative units cancel 15 of the 20 positive units, leaving 5 positive units ($^{+}5$), which is the *sum*. That is,

$$^{+}20 + {}^{-}15 = {}^{+}(20 - 15), \text{ or } {}^{+}5.$$

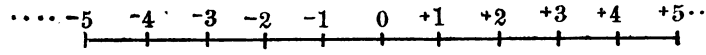
In (3) we see that 20 positive units cancel 20 of the 30 negative units, leaving 10 negative units ($^{-}10$), which is the *sum*. That is,

$$^{+}20 + {}^{-}30 = {}^{-}(30 - 20), \text{ or } {}^{-}10.$$

In (2) how is the *absolute value* of the sum found — by adding or by subtracting? Why has it the $+$ sign?

How is the absolute value of the sum found in (3)? Why is it negative? Which number has the greater absolute value? Upon what does the *quality* of the sum depend?

64. We may illustrate the adding of positive and negative numbers by representing them as standing on opposite sides of zero on a line laid off into units of length, thus:



1. The sum of $+2$ and $+3$ may be found by counting from $+2$ (whose distance from 0 is $+2$) 3 units to the *right*, or in the *positive direction*. The result is $+5$, the number of units from 0 to the right. How may the sum of -2 and -3 be found?

2. The sum of $+2$ and -3 may be found by counting 3 units to the *left*, or in the *negative direction*, from $+2$. It is -1. How may the sum of -2 and $+3$ be found?

Add the following:

1. -2 and $+4$.
2. -3 and $+3$.
3. -5 and $+2$.
4. -1 and $+5$.
5. $+4$ and -3.
6. $+3$ and -5.
7. $+5$ and 10.
8. -8 and $+12$.
9. $+4$ and -16.

65. A consideration of the preceding exercises and examples gives rise to the following method of finding the sum of two algebraic numbers:

1. *If the two numbers have like signs, add arithmetically their absolute values, and prefix to the result their common sign.*

2. *If they do not have the same sign, take the difference between their absolute values, and prefix to the result the sign of the one having the greater absolute value.*

Find the sums:

1. $+7 + +13$.
2. $-8 + +20$.
3. $-3 + -12$.
4. $+6 + -24$.
5. $-11 + +17$.
6. $+18 + -5$.
7. $-24 + +30$.
8. $+16 + -21$.
9. $+13 + -20$.
10. $-12.5 + +7.5$
11. $-15 + +30$.
12. $+25 + -25$.

What is the value of $a + b$:

13. When $a = {+6}$, $b = {-5}$?
14. When $a = {-6}$, $b = {+5}$?
15. When $a = 0$, $b = {+3}$?
16. When $a = 0$, $b = {-4}$?
17. When $a = {+7}$, $b = 0$?
18. When $a = {-9}$, $b = 0$?

19. A thermometer rises 50° above zero, then falls 60°. It then registers —— degrees below zero. Calling degrees below zero *negative*, express this algebraically.

20. A man travels 80 miles east, then 60 miles west. He is then —— miles from his starting point. If distances east be considered positive, how can these facts be expressed algebraically?

66. Sum of Several Numbers.

To find the algebraic sum of three or more numbers we add the second to the first, to this sum we add the third, and so on.

Thus, $a + b + c + d$ means $a + b$ with c added, and that sum with d added.

67. While in the preceding articles the operation of addition is taken to mean that the additions are to be performed from left to right, it is to be observed that *the result is the same in whatever order the numbers be added.* This is called the **Commutative Law** of addition.

Thus, $a + b \equiv b + a$, a and b being positive or negative numbers, just as $2 + 3 = 3 + 2$ in arithmetic.

This law may be proved as follows:

If a group of 3 objects is united with another group of 2 objects of the same kind, the number of the sum-group will be indicated by the symbols $2 + 3$.

If the group of 2 objects is removed, the group of 3 objects will remain. If the group of 2 objects is now united with the group of 3 objects, the number of the new sum-group will be indicated by the symbols $3 + 2$.

But the last sum-group contains the same objects as the first, hence the number in it has not been changed.

$$\therefore\ 2 + 3 = 3 + 2.$$

In any case the sum-group contains the *same things* as the individual groups contained, and this is true whatever the order of combination in which the different groups are brought together, and whatever the number of groups. Hence

$$+a + +b \equiv +b + +a.$$
$$+a + +b + +c \equiv +b + +a + +c,$$
$$\equiv +b + +c + +a,$$
$$\equiv +c + +a + +b,$$
$$\equiv \text{etc.}$$

The same law holds when some or all of the numbers are negative. For instance, in finding a man's cash balance, we can take into account the different items of *cash received* and *cash spent* in any order we please. The same number of positive units and negative units will cancel each other, and the same number will be left regardless of the order we follow.

68. It is also to be observed that the result is the same if any two or more of the numbers be associated or grouped in performing the additions. This is called the **Associative Law** of addition.

Thus, $a + b + c \equiv a + (b + c)$, a, b, and c representing positive or negative numbers, or both, just as in arithmetic $2 + 3 + 4 = 2 + (3 + 4) = 2 + 7$.

This law may be proved for *three* numbers, as follows:

Since $c + b + a \equiv (c + b) + a,$ (Art. 66)
$\equiv a + (c + b),$ (Com. Law)
$\equiv a + (b + c).$ (Com. Law)
$\equiv a + b + c.$ (Removing parentheses)

69. Zero in Addition.

If one of two numbers is zero, their algebraic sum is the other number.

Thus, $+3 + 0 = +3$.

The proof is as follows:

By Art. 66, $+3 + +1 + -1 = +4 + -1 = +3.$

But, by Associative Law,

$$+3 + +1 + -1 = +3 + (+1 + -1) = +3 + 0.$$

By Axiom 1, $+3 + 0 = +3.$

ADDITION OF MONOMIALS

70. The **degree of a monomial** is indicated by the number of its literal factors.

Thus, a^3 is one of the third degree, a^2b^3, of the fifth degree, etc.

The word *degree*, however, is usually limited to some particular letter; thus, while $2a^3x^2$ is of the fifth degree in a and x, it is said to be of the *second* degree in x.

71. Terms of the *same degree in each letter* are called **like terms**, and terms of the same degree in any particular letter are called *like terms* in that letter.

Thus, $2ax^2$ and $5ax^2$ are like terms in both a and x, and $2ax^2$ and $3a^2x^2$ are like terms in x.

72. The **degree of a polynomial** is determined by the highest degree of any of its terms.

Thus, $(ax^2 + 2bx + c)$ is of the second degree in x. Why?

73. As in arithmetic only like numbers can be added, so in algebra only like terms can be united by addition into one term.

Although unlike numbers cannot be united by addition into one term, an *indicated* operation is regarded as their algebraic sum. Thus, $m + n$ is called the sum of m and n. The sum of $+m$ and $+n$ may be indicated by $+(m + n)$, and the sum of $-m$ and $-n$ by $-(m + n)$.

1. Find the sum of $+2x$ and $+3x$.

$+2x$
$+3x$
$+5x$

Just as $+2 = +1 + +1$, so $+2x = +x + +x$; so also $+3x = +x + +x + +x$. Therefore, just as $2 + 3 = 5$, so $+2x + +3x = +(2+3)x = +5x$. Observe that the sum is obtained *by adding the coefficients of the like terms and annexing to the sum the common literal part.*

Add the following:

2.	3.	4.	5.	6.
$+4x$	$-x$	$5ab$	$-4my$	$(a+b)$
$+x$	$-3x$	$6ab$	$-2my$	$5(a+b)$
$+9x$	$-9x$	ab	$-7my$	$4(a+b)$

When numbers are positive, the symbol of quality $(+)$ is usually omitted, as in examples 4 and 6 above. When no symbol is written, $+$ is understood. The sign $-$ is never omitted.

7. Add $^{-}5\,xy$ to $^{+}2\,xy$.

$$^{+}2\,xy + {}^{-}5\,xy = {}^{-}(5-2)\,xy = {}^{-}3\,xy.$$

Add the following:

8. $^{+}8\,y$ to $^{-}6\,y$.
9. $^{-}12\,a$ to $^{+}5\,a$.
10. $^{+}8\,m$ to $^{-}3\,m$.
11. $^{+}3\,x$ to $^{-}12\,x$.
12. $18\,ab$ to $^{-}5\,ab$.
13. $^{-}24\,y$ to $^{+}30\,y$.
14. $3\,xy$ to $^{-}11\,xy$.
15. $8\,(m+n)$ to $^{-}7\,(m+n)$.
16. $^{-}(x+y)$ to $10\,(x+y)$.

17. Find the algebraic sum of $5\,x$, $^{-}7\,x$, $^{+}9\,x$, $^{-}4\,x$, and x.

The sum of the positive numbers is $15\,x$; that of the negative numbers is $^{-}11\,x$. These two sums united $= 4\,x$. What laws are made use of?

Add the following:

18.	19.	20.	21.
$3\,ax$	$^{-}9\,mn$	$34\,bcd$	$18\,(b-a)$
$^{-}5\,ax$	^{-}mn	$^{-}25\,bcd$	$^{-}36\,(b-a)$
$^{+}7\,ax$	$^{+}4\,mn$	$^{-}9\,bcd$	$^{-}50\,(b-a)$
$^{-}4\,ax$	$^{+}3\,mn$	$^{-}10\,bcd$	$^{-}50\,(b-a)$

22. Find the sum of $2\,ax$ and $3\,bx$.

$2\,ax$ and $3\,bx$ are like terms in x. The coefficients are $2\,a$ and $3\,b$, hence,

$$2\,ax + 3\,bx = (2\,a + 3\,b)\,x.$$

Express in the simplest form:

23. $8x + 3x + {}^{-}5\,x + x + {}^{-}4\,x + 12\,x + 7\,x$.

24. $15\,bx + {}^{-}6\,bx + bx + {}^{-}7\,bx + {}^{-}bx + 18\,bx + {}^{-}10\,bx$.

25. $5\,(m+n) + 13\,(m+n) + {}^{-}11\,(m+n) + 6\,(m+n) + {}^{-}20\,(m+n)$.

ADDITION OF POLYNOMIALS

74. To add polynomials we add their like terms, and express the sum in its simplest form. The unlike terms are written in succession with their proper signs.

1. Add $3x+y+^{-}2z$, $5x+^{-}4y+6z$, and $6y+^{-}9z+^{-}4x$.

(a)	(b)	*Check*
$3x+\ \ y+^{-}2z$	$3+\ 1+^{-}2$	$=\ 2$
$5x+^{-}4y+\ 6z$	$5+^{-}4+\ 6$	$=\ 7$
$^{-}4x+\ 6y+^{-}9z$	$^{-}4+\ 6+^{-}9$	$=\ ^{-}7$
$4x+\ 3y+^{-}5z$	$4+\ 3+^{-}5$	$=\ 2$

In (a), for convenience we write like terms in the same column. The sum of the first column is $+4x$, of the second $+3y$, and of the third $-5z$. Does it make any difference which column is added first? Why not?

If the arbitrary value 1 is substituted for each letter, is the value of each term its coefficient? Observe that the *value of the sum is equal to the sum of the value of the addends*, as shown in (b). Is this true for *any* value we may substitute for the letters? Does this *check* the work; that is, prove its accuracy?

Check the work in (a) by letting $x=1$, $y=2$, $z=3$.

When the terms involve many letters it is often convenient to add by using the *detached* coefficients, as in (b).

2. Add $5y+^{-}4c$, $3y+8c$, $5c+^{-}4y$, $y+^{-}c$, $y+^{-}10c$.

3. Add $6m+2n+^{-}5b$, $7n+^{-}3b+^{-}4m$, $b+8m+^{-}9n$, $m+5b$.

4. Add $3ab+5a^2b$, $6ab+^{-}8a^2b$, and $^{-}8ab+^{-}a^2b$.

5. Add $7a^2+^{-}2a+^{-}5$, and $^{-}9a^2+5a+3$.

6. Add $3x^2+y^2+^{-}3yz+^{-}z^2$, $2xy+^{-}3y^2+3yz$, and $^{-}4x^2+^{-}2xy+y^2+z^2$.

7. Add $a+2b+^{-}7c$, $7a+3b+^{-}5c$, and $3a+^{-}5b+3c$, and check the work by letting $a=1$, $b=1$, $c=1$.

8. Add $a^2+^{-}y+1$, $3+5y+^{-}2a^2$, $4y+^{-}2+7a^2$.

9. Add m^2+mn+n^2, $mn+^{-}3n^2+6m^2$, $8n^2+^{-}4m^2+^{-}2mn$.

10. Add $5b+^{-}4x^2+c$, $3b+^{-}c+7x^2$, x^2+b+c.

11. Show that if $x=^{-}3a+b+2c$, $y=2a+^{-}3b+c$, and $z=a+2b+^{-}3c$, then $x+y+z=0$.

If $A\equiv a^2+2ab+b^2$, $B\equiv a^2+^{-}2ab+b^2$, $C\equiv 2a^2+2ab+^{-}2b^2$, find the values of:

12. $A+B$.
13. $A+C$.
14. $B+C$.
15. $A+B+C$.

EXERCISES

75. 1. If a man has a sons, b daughters, and 1 wife, how many persons are in the family?

2. A boy who had 15 cents found m cents and earned $4\,m$ cents. How much had he then?

3. My house is d feet long and c feet wide. What is the distance around it?

4. Tom walked due east m hours, then due west n hours. If his rate was 3 miles an hour, at what distance from his starting point did he stop?

5. If m in the above problem is equal to n, where did Tom stop?

6. If $m = 5$ and $n = 3$, how far, and in what direction, from his starting point did he stop?

7. Locate his stopping place if $m = 4$ and $n = 6$.

8. D earns a dollars each week and spends b dollars. How much will he have at the end of 8 weeks?

9. What will be his financial condition if $a = \$15$ and $b = \$10$? What will it be if $a = \$12$ and $b = \$16$?

10. A thermometer indicated $+40°$; it then rose first $20°$, and then $30°$. What temperature did it then indicate?

11. A balloon is pulling upward with a force of 400 lb.; its two occupants weigh 150 lb. each. What is the combined weight? How is it expressed algebraically, if weight is considered positive?

12. A man has a house worth \$ 4000, and other property worth \$ 3500. He owes \$ 8000. Taking debt negatively, what represents his financial condition?

13. A man has \$ 500 and has debts to pay as follows: To A, \$ 100; to B, \$ 150, and to C, \$ 50. Show that he will have the same amount (\$ 200) left, whatever be the order of payment. What law is illustrated?

SUBTRACTION

76. 1. At 5 A.M. a thermometer registered 3° above zero ($^{+}3°$); at 7 A.M. it registered 8° above zero ($^{+}8°$). At 4 P.M. of the same day it registered 8° above zero, and at 6 P.M. 3° above zero.

In the first case, the difference in temperature was $^{+}5°$, and in the second case it was $^{-}5°$; that is, $^{+}8° - {}^{+}3° = {}^{+}5°$, and $^{+}3° - {}^{+}8° = {}^{-}5°$.

The positive result, $^{+}5°$, indicates that the mercury *rose* 5°; that is, moved in the *positive* direction from $^{+}3°$ to $^{+}8°$.

The negative result, $^{-}5°$, shows that the mercury *fell* 5°; that is, moved in the *negative* direction from $^{+}8°$ to $^{+}3°$.

$^{+}8° - {}^{+}3° =$ $\qquad$ $^{+}3° - {}^{+}8° =$

$^{+}8° + {}^{-}3° =$ $\qquad$ $^{+}3° + {}^{-}8° =$

2. A thermometer at 7 A.M. registered $^{-}6°$ (*i.e.*, 6° below 0); at 12 M. it registered $^{+}9°$ (*i.e.*, 9° above 0); and at 5 P.M. it again registered $^{-}6°$. What was the rise in the forenoon? The fall in the afternoon?

$^{+}9° - {}^{-}6° =$ $\qquad$ $^{-}6° - {}^{+}9° =$

$^{+}9° + {}^{+}6° =$ $\qquad$ $^{-}6° + {}^{-}9° =$

From 6° below zero to 9° above zero is how many degrees in the positive direction? From 9° above to 6° below is 15° in which direction? Does adding a positive number give the same result as subtracting an equal negative number?

3. If to-day a thermometer registers $^{-}4°$, and yesterday it registered $^{-}8°$, how much warmer is it to-day than it was yesterday? That is, $^{-}4° - {}^{-}8° = ?$ How much colder was it yesterday than to-day? That is, $^{-}8° - {}^{-}4° = ?$

$^{-}4° - {}^{-}8° =$ $\qquad$ $^{-}8° - {}^{-}4° =$

$^{-}4° + {}^{+}8° =$ $\qquad$ $^{-}8° + {}^{+}4° =$

4. What is the remainder after subtracting $^-3$ from $^-10$? After subtracting $^-10$ from $^-3$? What number added to $^-3$ gives $^-10$? What number added to $^-10$ gives $^-3$?

77. In algebra, as in arithmetic, the *minuend* is the sum of the subtrahend and the remainder (difference). As an operation **Subtraction** is the *inverse* of addition.

Thus, $10 + {}^-4 = 6$, the sum, which we may regard as a minuend.
Taking 10 as the subtrahend, we have $6 - {}^+10 = {}^-4$, the remainder.
Taking $^-4$ as the subtrahend, we have $6 - {}^-4 = {}^+10$, the remainder.
In either case 6 is the sum of 10 and $^-4$.

Carefully examine the following:

$8 - 2 = 6.$	$^-11 - {}^+2 = {}^-13.$	$1 - 2 = {}^-1.$
$8 + {}^-2 = 6.$	$^-11 + {}^-2 = {}^-13.$	$1 + {}^-2 = {}^-1.$
$8 = 6 + 2.$	$^-11 = {}^-13 + {}^+2.$	$1 = {}^-1 + 2.$

Observe that subtracting $^+2$ is equivalent to adding $^-2$; also, that the

Minuend = Remainder + Subtrahend.

Examine these equations:

$7 - {}^-4 = 11.$	$^-12 - {}^-4 = {}^-8.$	$2 - {}^-4 = 6.$
$7 + 4 = 11.$	$^-12 + 4 = {}^-8.$	$2 + 4 = 6.$
$7 = 11 + {}^-4.$	$^-12 = {}^-8 + {}^-4.$	$2 = 6 + {}^-4.$

Observe that subtracting $^-4$ is equivalent to adding $^+4$; and again that the

Minuend = Remainder + Subtrahend.

ILLUSTRATION I

A man whose income is \$100 a month spends \$60, and saves \$40. If his income is reduced \$10 a month, he will save \$30.

$$\$90 - \$60 = \$30.$$

Or, if his expenses are increased \$10 a month, he will save \$30.

$$\$100 - \$70 = \$30.$$

Hence, to *take away* \$10 *income* is equivalent to *adding* \$10 *expenses.* Either reduces his savings to \$30.

Calling income and savings *positive,* and expenses *negative,* we have the following algebraic expression of this relation:

$$\$40 - {}^{+}\$10 = \$30.$$

$$\$40 + {}^{-}\$10 = \$30.$$

$$\therefore \quad \$40 - {}^{+}\$10 = \$40 + {}^{-}\$10. \qquad \text{(Why ?)}$$

ILLUSTRATION II

If his income is increased \$10 a month, he will save \$50.

$$\$110 - \$60 = \$50.$$

Or, if his expenses are reduced \$10 a month, he will save \$50.

$$\$100 - \$50 = \$50.$$

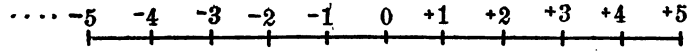
Hence, to *take away* \$10 *expenses* is equivalent to *adding* \$10 *income.* Either increases his savings to \$50. The relation is algebraically expressed thus:

$$\$40 - {}^{-}\$10 = \$50.$$

$$\$40 + {}^{+}\$10 = \$50.$$

$$\therefore \quad \$40 - {}^{-}\$10 = \$40 + {}^{+}\$10. \qquad \text{(Why ?)}$$

78. These results are nicely illustrated by reference to the representation of positive and negative numbers as standing on opposite sides of zero as given in Art. 64.



1. The result of subtracting $^{+}2$ from $^{+}3$ is found by counting from $^{+}3$ (whose distance from 0 is 3 positive units) 2 units to the left, or in the *negative* direction; it is $^{+}1$, the number of units from 0 to $^{+}1$. How may the result of subtracting $^{+}2$ from $^{-}3$ be found?

2. The result of subtracting $^-2$ from $^+3$ is found by counting from $^+3$ two units to the right, or in the positive direction; that is, in a direction *opposite* to that indicated by the sign $^-$ belonging to 2; it is $^+5$, which means that the distance from $^-2$ to $^+3$ is 5 units in the positive direction. From $^+3$ to $^-2$ is 5 units in the negative direction; hence $^-2 - {}^+3 = {}^-5$.

3. In arithmetic the difference between two numbers is the remainder found by subtracting the less from the greater. In algebra, however, it is the remainder found by subtracting *either* from the other; it is the number which, added to the subtrahend, will produce the minuend. Hence, whether the difference is positive or negative depends upon which of the two numbers is regarded as the minuend. This is illustrated by the following examples. Verify them by reference to the diagram.

Minuend	$^+7$	$^+2$	$^-7$	$^-2$	$^+7$	$^-2$	$^-7$	$^+2$
	$^+2$	$^+7$	$^-2$	$^-7$	$^-2$	$^+7$	$^+2$	$^-7$
Difference, or Remainder	$^+5$	$^-5$	$^-5$	$^+5$	$^+9$	$^-9$	$^-9$	$^+9$

4. It appears from the preceding that the remainder is *less* than the minuend when the subtrahend is *positive*, and *greater* than the minuend when the subtrahend is *negative*. In arithmetic the remainder is never greater than the minuend.

79. From the preceding exercises and illustrations we derive the following:

The subtraction of any algebraic number is equivalent to the addition of the equal opposite number; or, which is the same thing, **the addition of the number with its sign of quality reversed.**

80. The formal proof of the principle involved may be given as follows:

1. To subtract $^+2$ from $^+5$ is to find a number such that if $^+2$ be added the sum will be $^+5$. The result may be written $^+5 - {}^+2$.

Then, since by definition of subtraction, remainder plus subtrahend equals minuend, we have

$$(^{+}5 - {}^{+}2) + {}^{+}2 = {}^{+}5,$$

$$^{+}5 - {}^{+}2 + {}^{+}2 + {}^{-}2 = {}^{+}5 + {}^{-}2, \qquad \text{(by adding } {}^{-}2 \text{ to each side)}$$

$$^{+}5 - {}^{+}2 + 0 = {}^{+}5 + {}^{-}2, \qquad \text{(since } {}^{+}2 + {}^{-}2 = 0\text{)}$$

or $$^{+}5 - {}^{+}2 = {}^{+}5 + {}^{-}2. \qquad \text{(Art. 69)}$$

That is, to *subtract* $^{+}2$ is equivalent to *adding* $^{-}2$.

Again, to subtract $^{-}2$ from $^{+}5$, we have

$$(^{+}5 - {}^{-}2) + {}^{-}2 = {}^{+}5,$$

$$^{+}5 - {}^{-}2 + {}^{-}2 + {}^{+}2 = {}^{+}5 + {}^{+}2, \qquad \text{(by adding } {}^{+}2\text{)}$$

$$^{+}5 - {}^{-}2 = {}^{+}5 + {}^{+}2. \qquad \text{(since } {}^{-}2 + {}^{+}2 = 0\text{, and Art. 69)}$$

That is, to *subtract* $^{-}2$ is equivalent to *adding* $^{+}2$.

2. In general, it is to be proved that

$$m - {}^{-}s \equiv m + {}^{+}s.$$

Let m be any minuend, positive or negative, and ^{-}s any negative subtrahend. Then $(m - {}^{-}s)$ must be such a number that

$$(m - {}^{-}s) + {}^{-}s \equiv m. \qquad \text{(by definition of subtraction)}$$

$$m - {}^{-}s + {}^{-}s + {}^{+}s \equiv m + {}^{+}s, \qquad \text{(by adding } {}^{+}s \text{ to each side)}$$

$$m - {}^{-}s + 0 \equiv m + {}^{+}s, \qquad \text{(since } {}^{-}s + {}^{+}s = 0\text{)}$$

$$m - {}^{-}s \equiv m + {}^{+}s. \qquad \text{(Art. 69)}$$

In like manner,

$$m - {}^{+}s \equiv m + {}^{-}s.$$

Have the pupil write out the proof.

81. We are thus naturally led to the following convenient methods of subtracting — the first growing directly out of the definition of subtraction, the second being established by the foregoing proofs:

1. *Find the number which added to the subtrahend gives the minuend.*

2. *Reverse the sign of the subtrahend, and add.*

1. From $4\,x + {}^{-}3\,y$ take $x + 2\,y$.

$$\begin{array}{r} 4\,x + {}^{-}3\,y \\ x + 2\,y \\ \hline 3\,x + {}^{-}5\,y \end{array}$$

What term added to x gives $4\,x$?
What term added to $2\,y$ gives $-3\,y$? $(-5\,y.)$
Or, $4\,x + {}^{-}x = 3\,x$; ${}^{-}3\,y + {}^{-}2\,y = {}^{-}5\,y$.
Changing the sign of the subtrahend should be done mentally.

Subtract:

2. $3x$ from $8x$.
3. $13y$ from $^{-}17y$.
4. $^{-}5a$ from $^{-}12a$.
5. $^{-}6b$ from $4b$.
6. $7x$ from $2x$.
7. $6x + ^{-}2y$ from $8x + ^{-}5y$.
8. $9ab + 3d$ from $12ab + ^{-}d$.
9. $a + ^{-}b$ from $a + b$.
10. $x + ^{-}y + 0$ from $x + y + z$.
11. $m + n + ^{-}a$ from $3m + ^{-}4n + 0$.

82. One Set of Signs for Quality and Operation.

From a consideration of Art. 64, it is seen that adding positive units to any number is *in effect* but counting them *on* (*i.e.*, in the positive direction from that number); and that adding negative units to any number is but counting them *off* (*i.e.*, in a negative direction from that number). From this illustration of positive and negative numbers, as well as from other explanations of them, we find that

$^{+}1$ may be regarded as $0 + 1$, and $^{-}1$ as $0 - 1$.

$^{+}3$ may be regarded as $0 + 3$, and $^{-}3$ as $0 - 3$.

Hence, in addition, we may make one set of signs serve to denote both *quality* and *operation;* and since every operation of subtraction is equivalent to one of addition, it is evident that the one set of signs will suffice also in subtraction.

83. Hitherto the two sets of signs have been used to keep prominent the distinction between *quality* and *operation.* It is customary and convenient, however, to use only the one set of signs for both purposes, and this practice we shall hereafter follow, except in cases where clearness is gained by a retention of the double set.

When needed to avoid ambiguity, the numbers and their signs of quality may be inclosed in parentheses.

Thus, if $x = 1$, $y = 2$, $z = ^{-}3$, then $x + y + z = 1 + 2 + (-3)$.

84. As before stated, the sign $+$ is usually omitted when it denotes quality, but the sign $-$ is never omitted. It should be

remarked that by the *sign of a term* is meant the + or − sign that denotes the *quality* of the term. We may now summarize results as follows:

(*a*) $^{+}2+{}^{+}3=2+(+3)=2+3$, by omitting the sign of *quality*.

(*b*) $^{+}2+{}^{-}3=2+(-3)$, in which + indicates *operation* and − denotes *quality*.

(*c*) $^{+}2-{}^{+}3=2-(+3)=2-3$, by omitting the sign of *quality*.

(*d*) $^{+}2-{}^{-}3=2-(-3)$, in which the − within the parentheses denotes *quality*, while the one preceding indicates operation.

(*e*) $^{+}2+{}^{-}3-{}^{+}5-{}^{-}4=(+2)+(-3)-(+5)-(-4)$, the signs *without* the parentheses indicating *operation*, those *within* denoting *quality*.

In (*b*), $2+(-3)=2-(+3)=2-3=-1$.

In (*d*), $2-(-3)=2+(+3)=2+3=5$.

In (*e*), $(+2)+(-3)-(+5)-(-4)=(+2)-(+3)-(+5)+(+4)$, or $2-3-5+4$ (dropping signs of quality) $=-2$.

QUERY. Does the last sign denote quality or operation? Why?

SUBTRACTION OF POLYNOMIALS

85. Like terms can be united by subtraction into a single term, but unlike terms are subtracted by writing them in succession with the sign − before each term to be subtracted.

Thus, to subtract a and b from c, we write $c-a-b$. To subtract $-n$ from m, we write $m-(-n)$, which is identical with $m+n$.

1. Subtract $3\,ax-8\,y+5\,b$ from $6\,ax-4\,y-3\,b$.

(1)	(2)
$6\,ax-4\,y-3\,b$	$6-4-3=-1$
$3\,ax-8\,y+5\,b$	$3-8+5=\ \ 0$
$3\,ax+4\,y-8\,b$	$3+4-8=-1$

Since $(6\,ax-4\,y-3\,b)-(3\,ax-8\,y+5\,b)\equiv 3\,ax+4\,y-8\,b$, the identity is true for any values of a, b, x, and y. Hence, if the value 1 be substituted for each letter, the value of each term will be its numerical coefficient, and the subtraction is readily performed by *detaching* the coefficients and subtracting each term separately as in (2). How is the work checked?

NOTE. The method of *detached coefficients* is much to be preferred in both additions and subtractions in examples in which the terms involve many letters.

2. From $a^2 + 2\,ab + b^2$ subtract $a^2 - 2\,ab + b^2$.

3. Subtract $4\,c(b-d) + 6\,am$ from $10\,c(b-d) + 6\,am$.

4. From $3\,x^2 + 2\,xy + y^2$ take $x^2 - xy - y^2$.

5. Subtract $7\,x^2y - 5\,ab + 2\,m$ from $4\,x^2y - 3\,ab + 5\,m$.

Check by adding the remainder to the subtrahend.

6. Subtract $m - 4\,n - p + 8$ from $m - 3\,n + p - 7$.

Check by letting $m = 3$, $n = 1$, $p = 2$.

7. Subtract $\frac{2}{3}\,a + \frac{1}{2}\,b - \frac{1}{6}\,c$ from $\frac{3}{2}\,a - \frac{3}{4}\,b + \frac{2}{3}\,c$.

Use method of detached coefficients.

8. From $a^3 - 2\,a^2 - 1$ take $a^2 - b + 1$.

9. What expression must be added to $a^2 - 9\,ab + 7\,b^2$ to make $4\,a^2 - 5\,ab + 2\,b^2$?

10. From $x - y - z$ take $x - y - 1$.

11. Take $5(m+n) - a + 2$ from $8(m+n) + a$.

12. From $7\,abc - 3\,a^2 + 5\,b^2 - c^2$ take $a^2 + b^2 + c^2 - 3\,abc$.

How are the terms conveniently arranged before subtracting?

13. Subtract $1 + x - x^2 + x^3 - x^4$ from $2 - x - x^2 - x^3 + x^5$.

14. Subtract $2\,x^m - 3\,x^n + x$ from $5\,x^m + x^n + x$.

15. What expression must be subtracted from $x^2 - x + 1$ that the remainder may be x^3?

16. From $a + b + c$ subtract the sum of $a - b - c$, $b - c - a$, and $c - a - b$.

17. Subtract the sum of $x^2 + 2\,xy + y^2$ and $2\,x^2 + 2\,y^2$ from $x^2 + 2\,xy + y^2$.

18. The subtrahend is $a^2 - 3\,ab - 5$; the remainder is $a^2 + 3\,ab + 5$. What is the minuend?

19. What expression must be added to $m^2 - n^2$ to produce 0?

20. Show that the difference between $10\,x$ and $-7\,x$ is $-17\,x$. What is it when $10\,x$ is taken as the minuend?

21. From the sum of $2x - 5 + 7x^2$ and $3x + 4 - 2x^2$ take $3x - 1 + 5x^2$.

Use method of detached coefficients; reverse signs of subtrahend and add the three polynomials.

22. If $A \equiv a^2 - 2ab + b^2$, $B \equiv 2a^2 - ab + b^2$, and $C \equiv 2a^2 + 2b^2$, find the expression for:

(*a*) $A - B$. (*c*) $B - C$. (*e*) $A + C - B$.

(*b*) $B - A$. (*d*) $A + B - C$. (*f*) $A - B - C$.

Find by trial which is the best method of solving (*d*), (*e*), and (*f*).

SYMBOLS OF AGGREGATION

86. We have already seen that parentheses and other symbols of aggregation are used to show that an inclosed expression is to be treated as a single number.

Thus, $7 + (8 + 5)$ means that the sum of 8 and 5, or 13, is to be added to 7. What does $7 + (8 - 5)$ mean? Would the value of the expression be the same if the parentheses were removed?

The expression $8 - (4 + 3)$ means that $4 + 3$, or 7, is to be subtracted from 8.

$$8 - 7 = 1.$$

If we remove the parentheses, we have

$$8 - 4 + 3 = 7.$$

The removal has changed the value of the expression.

87. Parentheses may be removed without changing the value of the expression. The principles governing the removal of symbols of aggregation result immediately from the principles of addition and subtraction, and the application of the Associative Law.

1. We have $7 + (2 + 3) = 7 + 2 + 3$, since to add the sum $2 + 3$, or 5, is equivalent to adding 2 and 3 in succession. In general,

$$a + (b + c) \equiv a + b + c. \quad \text{(by Associative Law)}$$

Again, $7 + (2 - {}^+3) = 7 + (2 + \ 3)$, since to add $^-3$ is equivalent to subtracting $^+3$.

Removing parentheses, $7 + 2 - {}^{+}3 = 7 + 2 + {}^{-}3 = 7 + 2 - 3$.

In general, $a + (b - c) \equiv a + b - c$. That is, the removal of parentheses preceded by the sign of addition does not change the value of the expression.

The foregoing examples illustrate the following principle:

(*a*) **A symbol of aggregation preceded by the sign of addition, +, may be removed, and the sign of each term within be left unchanged.**

2. We have $7 - (2 + 3) = 7 - 2 - 3$, since to subtract the sum of $2 + 3$, or 5, is equivalent to subtracting 2 and 3 in succession. In general,

$$a - ({}^{+}b + c) \equiv a - b - c.$$

Again, $7 - (2 - {}^{+}3) = 7 - (2 + {}^{-}3)$, since to add $^{-}3$ is equivalent to subtracting $^{+}3$.

Removing parentheses, $7 - 2 + {}^{+}3 = 7 - 2 - {}^{-}3 = 7 - 2 + 3$.

In general, ${}^{+}a - ({}^{+}b - {}^{+}c) \equiv {}^{+}a - {}^{+}b + {}^{+}c$. That is, the removal of parentheses preceded by the sign of subtraction changes the value of the expression, which is avoided by changing the sign of each inclosed term.

The preceding examples illustrate the following principle:

(*b*) **A symbol of aggregation preceded by the sign of subtraction, −, may be removed, provided the sign of each term within be reversed.**

Remove parentheses and find values of:

1. $18 + (7 + 3)$.
2. $21 + (9 - 6)$.
3. $30 - (20 + 5)$.
4. $30 - (20 - 5)$.
5. $30 - ({}^{-}20 + {}^{-}5)$.
6. $30 - ({}^{-}20 + {}^{+}5)$.
7. $30 - ({}^{-}20 - {}^{-}5)$.
8. $30 - ({}^{-}20 - {}^{+}5)$.
9. $25 - (10 - 5 - 8)$.
10. $17 - (3 + 12 - 6)$.
11. $a - (b + c)$, when $a = 10$, $b = 3$, $c = 5$.
12. $m - (n - a)$, when $m = 16$, $n = 16$, $a = 16$.

Remove the parentheses in the following, then unite like terms in each result:

13. $m + 2mn + (m - n + p)$.
14. $3y + 17x - (y + 16x)$.
15. $5(m + n) - 7xy - (m + n) + 4xy$.
16. $x - \{x - [x - (2x - x)]\}$.
17. $(a + b + c) - (a + b + c)$.
18. $5a - (3b - 2a) + 4b - (6a - 8b)$.
19. $12(a + b + c) - 8(a + b + c) - 4b + 20c$.

88. When two or more sets of symbols of aggregation are employed, they may be removed in succession by applying the principles already learned. Either the *inmost* or the *outmost* may be first removed, preferably the latter, except where there is a coefficient before the parentheses.

Remove the symbols of aggregation in the following:

1. $a+2b-\{6a-[3b-(6a-6b)]\}$.

(a)	(b)
$a+2b-\{6a-[3b-(6a-6b)]\}$,	$a+2b-\{6a-[3b-(6a-6b)]\}$,
$a+2b-6a+[3b-(6a-6b)]$,	$a+2b-\{6a-[3b-6a+6b]\}$,
$a+2b-6a+3b-(6a-6b)$,	$a+2b-\{6a-3b+6a-6b\}$,
$a+2b-6a+3b-6a+6b$,	$a+2b-6a+3b-6a+6b$,
$11b-11a$.	$11b-11a$.

In beginning with the outer symbol, as in (a), how many changes of sign were made in the successive removals? How many in removing the inner symbol first, as in (b)? In this respect, which method is to be preferred?

Simplify the following:

2. $a - [2ab - (b^2 - a^2) + b^2]$.
3. $m - [(a - b) - (c - m)]$.
4. $2a - \{3b - [2a - (3b - 2a)]\}$.
5. $a - 2b - [3a - (b - c) - 5c]$.

6. $(4x+3y)-\{3y-[4z+(m-7x)]\}$.
7. $(m-n)-[-m-(n-m)+(m-n)]$.
8. $x^3-[x^2-(1-x)]-\{1+[x^2-(1-x)+x^3]\}$.
9. $(m-n+p)-(m-n-p)+(m+n-2p)$.
10. $x-\{2x-[3x-4x-1]\}-[3x-(1+3x)]$.
11. $3m-n-[p-(2m-7n)]-[4m-(7n-p)]$.
12. $5y+3d-[4x-(3y+2d-1)]-[8y-(4x+6d)]$.

89. Since the insertion of symbols of aggregation is the converse of the operation of removing them, any terms of a polynomial may be inclosed within symbols of aggregation preceded by the sign of addition; and any terms may be inclosed within symbols of aggregation, preceded by the sign of subtraction, provided the sign of each term within be reversed.

Thus, $a+b-c+d \equiv a+(b-c+d)$
$\equiv a+b+(-c+d)$,

and $a-b+c-d \equiv a-(b-c+d)$
$\equiv a-b-(-c+d)$.

Insert parentheses to inclose:

1. x and y in the expression $a-x+b-y$.
$a+b-(x+y)$, or $a+b+(-x-y)$.
2. m and n in the polynomial $x+m-b-n$.
3. a and b in the expression $c+d-a+b$.
4. y^2 and z^2, and y and z in the expression z^2-z+y^2-y.
5. m^2, n^2, and $2mn$ in the polynomial $3x-m^2+n^2-2mn$.

EXERCISES

90. 1. A has \$10; B has no money and is \$5 in debt. How much more is A worth than B?

2. Tom has \$50 in bank. Harry has \$20 in cash, but owes John \$40. Tom is worth how much more than Harry?

3. Mary has 8 jacks, and Alice has $^{-}5$ (*i.e.*, owes 5). Alice has how many fewer than Mary?

4. In the schoolroom the temperature is 70° above zero, while outside it is 5° below zero. How many degrees warmer is it inside than outside? $70° - {}^{-}5° = (\ )$.

5. A is \$35 in debt, and B is \$50 in debt. How much better off is A than B? $\$35 - {}^{-}\$50 = (\ )$.

6. By selling a cow for a dollars I gained b dollars. What did the cow cost?

7. A and B together have n children, of whom A has 10. How many has B?

8. A man who earns d dollars a month spends p dollars for rent and q dollars for other purposes. How much does he save in a month?

9. A boy wishing to ride n miles on his wheel rode a miles the first day and b miles on each of the next two days. How many miles had he yet to ride?

10. Make an example showing the advantage of using detached coefficients in additions and subtractions.

11. Make an example showing the advantage of beginning at the outside in removing symbols of aggregation. Try to write the final result at sight.

12. Having the expression $2a + 4b - 3c + 2d + 5e$, inclose any two terms after the first within symbols of aggregation.

13. Place within parentheses the like powers of x in $x^5 + x^3 - 2x^4 - 3x^3 + ax^5 - bx^4 + cx^3$.

14. Show that $a - b - (c - d + e) \equiv a - b - c + d - e$.

15. Since $2 - ({}^{+}3) = 2 - 3$, we have, by subtracting 2 from each side, $0 - ({}^{+}3) = 0 - 3 = {}^{-}3$. What is the result when $^{-}3$ is subtracted from 0?

16. What is the difference between the net capital of A, who has cash in bank \$1000, land worth \$5000, and owes \$3000, and that of B who has land worth \$9000 and owes \$4000? Express algebraically the excess of B's capital over A's.

MULTIPLICATION

91. In a certain factory it was found that there was a *gain* to the business of $2 a day on each additional operative employed, and a *loss* of $2 a day on each person added to the office force.

Considering persons added and gains as *positive*, and the opposites as *negative*, give the meaning of each of the following:

1. If five additional operatives are employed, what are they worth a day to the business?

$$+\$2 \times (+5) = +\$10.$$

2. If five of the operatives fail to work for one day, what is the change worth to the business? In men? In money?

$$+\$2 \times (-5) = -\$10.$$

3. If five persons are added to the office force, what are they worth each day to the business?

$$-\$2 \times +5 = -\$10.$$

4. If five of the latter force are now dropped, what is the daily gain to the business? In men? In money? How is it expressed algebraically?

$$-\$2 \times -5 = +\$10.$$

5. When both factors have the same sign, what is the sign of the product?

6. What is the sign of the product of two factors having unlike signs?

7. What is the sign of the product of 4 factors if two of them are positive and two negative?

8. If three factors are negative and one positive, what is the sign of their product? If one is negative and three positive?

9. When the number of negative factors is *odd*, is the sign of the product positive or negative? Show by examples.

92. Multiplication is the operation of taking a number (the *multiplicand*) as many times as there are units in another number (the *multiplier*), the result being called the *product*.

In arithmetic, this definition originally applied only to operations in which the multipliers were integers; in later years, however, it was extended to include fractional multipliers. In algebra, there has been a further extension to include *all* multipliers.

93. Zero in Multiplication.

If one of two or more factors is 0, *the product is* 0.

Thus, just as $1 \times 2 = 1 + 1$, so $0 \times 2 = 0 + 0$; and just as $2 \times 3 = 3 \times 2$, so $a \times 0 = 0 \times a = 0$. That the order of factors may be changed is shown hereafter.

94. When the multiplier is *positive*, the process of multiplication in algebra is the same as that in arithmetic, except that the sign of quality of the multiplicand is to be written before the product. Thus:

$$(a) \quad {}^{+}5 \times {}^{+}3 = {}^{+}5 + {}^{+}5 + {}^{+}5 = {}^{+}15.$$
$$(b) \quad {}^{-}5 \times {}^{+}3 = {}^{-}5 + {}^{-}5 + {}^{-}5 = {}^{-}15.$$

Observe that 3 times 5 positive units gives 15 positive units as a product, and 3 times 5 negative units gives 15 negative units as a product.

95. When the multiplier is *negative*, it gives to the product a sign opposite to that given by a positive multiplier; that is, the quality of the multiplicand is reversed in the product. Note the following:

$$(c) \quad {}^{+}5 \times {}^{-}3 = {}^{-}15.$$
$$(d) \quad {}^{-}5 \times {}^{-}3 = {}^{+}15.$$

Since 3 is equal to (say) $5 - 2$, to multiply by 3 is the same as to multiply by $(5 - 2)$; that is, to multiply by 5, then by 2, and subtract the latter product from the former. Thus, $20 \times 3 = 20 \times 5 - 20 \times 2 = 100 - 40 = 60$. Similarly, since $^{+}3 + {}^{-}3 = 0$, $^{-}3 = 0 - {}^{+}3$; hence to multiply by $^{-}3$, we may multiply first by 0, then by $^{+}3$, and subtract the latter product from the former. But the former product is 0, and in subtracting the latter product from 0 we *reverse* the sign of the subtrahend.

Hence in (*c*), $^{+}5 \times {}^{-}3$ may be interpreted to mean that $^{+}5$ is to be taken three times, and then the result *reversed in quality* (subtracted from 0), giving the product the sign opposite to that of the multiplicand.

Similarly in (*d*), $^{-}5 \times {}^{-}3$ may be regarded as signifying that $^{-}5$ is to be taken 3 times, and the sign of the result ($^{-}15$) reversed, $^{-}15$ thus becoming $-(^{-}15)$, or $^{+}15$.

QUERIES. 1. In (*a*) and (*b*) the product has the *sign* of the multiplicand. Is this true in (*c*) and (*d*), where the multiplier is negative?

2. What is the sign of the product when both factors are positive, as in (*a*)? When both are negative, as in (*d*)? When they have unlike signs, as in (*b*) and (*c*)?

96. If a and b stand for any two numbers, we have

$$^{+}b \times {}^{+}a \equiv {}^{+}(ba),$$
$$b \times {}^{+}a \equiv {}^{-}(ba),$$
$$^{+}b \times {}^{-}a \equiv {}^{-}(ba),$$
$$^{-}b \times {}^{-}a \equiv {}^{+}(ba).$$

The preceding examples exhibit the following

LAW OF SIGNS. **When two factors have like signs, the product is positive; when they have unlike signs, the product is negative.**

ILLUSTRATIONS

A train whose speed is 20 miles an hour runs north and south, passing a point P at 12 M. Locate the train at 5 P.M. and at 7 A.M.

Consider as *positive:* (1) distances north; (2) train's rate northward; (3) time after 12 M.

Consider as *negative:* (1) distances south; (2) train's rate southward; (3) time before 12 M.

Then, if the train is running northward,

(*a*) in 5 hours after 12 M. it will be 100 miles north of *P*, which is expressed algebraically by

$$+20 \times +5 = +100.$$

(*b*) 5 hours before 12 M. it will be 100 miles south of *P*, expressed algebraically by

$$+20 \times -5 = -100.$$

If the train is running southward,

(*c*) in 5 hours after 12 M. it will be 100 miles south of *P*, expressed algebraically by

$$-20 \times +5 = -100.$$

(*d*) 5 hours before 12 M. it will be 100 miles north of *P*, expressed algebraically by

$$-20 \times -5 = +100.$$

Notice that we never multiply *miles* by *hours*, but numerical value by numerical value.

Multiply the following:

1. $4a$ by 3.
2. $-b$ by 7.
3. $-5x$ by 2.
4. $-6y$ by $-d$.
5. $-8m$ by -1.
6. $9xy$ by $-a$.
7. $-ab$ by 10.
8. $-c^2d$ by -5.
9. -12 by $2x$.
10. -1 by $15p$.

NOTE. Since in its primitive meaning multiplication may be regarded as shortened addition, it follows that, as in addition, but one set of signs is required to denote both quality and operation.

97. Continued Products. The product of three factors is interpreted to mean the product of the first and second multi-

plied by the third; the product of four factors means the product of the first and second multiplied by the third, and that product by the fourth; and so on.

Thus, abc means ab multiplied by c; $abcd$ means ab multiplied by c, and that product (abc) by d.

98. When the number of negative factors is *odd*, the product is *negative;* when *even*, the product is *positive*. Thus,

$a \times (-b) \times c \equiv -abc.$ $\quad b \times b \times (-c) \times (-c) \equiv b^2c^2.$

$a \times (-b) \times (-c) \equiv abc.$ $\quad b \times (-b) \times (-c) \times (-c) \equiv -b^2c^2.$

99. The Commutative Law of Multiplication.

In arithmetic, $2 \times 3 = 3 \times 2$. It will now be shown that in algebra $ab \equiv ba$, a and b being any positive integers.

Suppose blocks are arranged as in Fig. 1. Then, since there are 6 blocks in the row AB, and there are 4 such rows, there are (6×4) blocks.

But since there are 4 blocks in the row AD, and there are 6 such rows, there are (4×6) blocks.

Evidently the number of blocks in each case is the same.

$\therefore\ 6 \times 4 = 4 \times 6.$

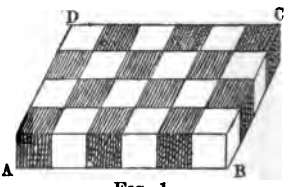


FIG. 1.

And since the proof is independent of the particular numbers, 4 and 6, we have in general,

$$ab \equiv ba.$$

For extension of this proof see the following article.

100. The Associative Law of Multiplication.

The factors of a product may be taken in any desired order, and may be grouped (associated) in any convenient way without affecting the value of the product.

Suppose blocks are built up as in Fig. 2. Then, since there are

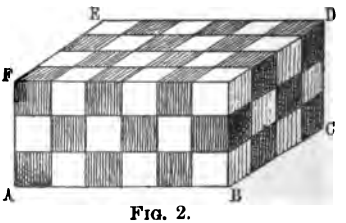


FIG. 2.

6 blocks in the row AB, and there are 4 such rows, there are (6×4) blocks in one layer, and in 3 layers there are $(6 \times 4 \times 3)$ blocks.

Again, let BD be regarded as the base. Then, since there are 4 blocks in the row BC, and there are 3 such rows, there are (4×3) blocks in that layer, and in the 6 layers there are $(4 \times 3 \times 6)$ blocks.

Similarly, if BF is regarded as the base, there are $(6 \times 3 \times 4)$ blocks in all; and so on.

Evidently the number of blocks in each case is the same;

$$\therefore\ 6 \times 4 \times 3 = 4 \times 3 \times 6 = 6 \times 3 \times 4 = \cdots.$$

And since the proof is independent of the particular numbers, we have in general,

$$abc \equiv bca \equiv acb \equiv \cdots.$$

Moreover, since in multiplying by negative numbers we perform the operations as if the numbers were positive, giving the products the proper sign, the foregoing proof is sufficiently general for any integers.

It is seen from the preceding that the Commutative Law and the Associative Law may be applied simultaneously in the multiplication of numbers.

NOTE. Since the existence of these two laws removes the necessity for maintaining a strict order of writing the multiplier and multiplicand in the product, we begin to think of co-factors, and call any one of them or any association of them a coefficient of the other factor or factors. The numerical coefficient is written arbitrarily first in a product. Notice that the coefficient is really a multiplier. Compare Art. 17.

101. The Law of Exponents.

As we have seen, a^2 means aa, and a^3 means aaa. Therefore,

$$a^2 \times a^3 \equiv aa \times aaa,$$
$$\equiv aaaaa, \quad \text{(by Associative Law)}$$
$$\equiv a^5. \quad \text{(by definition of exponent)}$$

If m and n are positive integers, we have in general,

$$a^m \equiv aaa \cdots \text{ to } m \text{ factors},$$

and $$a^n \equiv aa \cdots \text{ to } n \text{ factors}.$$

$$\therefore\ a^m \times a^n \equiv (aaa \cdots \text{ to } m \text{ factors}) \times (aa \cdots \text{ to } n \text{ factors}),$$
$$\equiv aaaaa \cdots \text{ to } (m+n) \text{ factors}, \quad \text{(by Associative Law)}$$
$$\equiv a^{m+n}.$$

Thus, $$2\,ab^2c^3 \times 5\,a^3bc^2 \equiv 10\,a^4b^3c^5.$$

In a similar manner the law can be extended to the case of three or more factors.

Thus, $a^m \times a^n \times a^p \equiv a^{m+n+p}$; for $a^m \times a^n \times a^p \equiv (a^m \times a^n) \times a^p \equiv a^{m+n} \times a^p \equiv a^{m+n+p}$.

In the preceding examples the exponents have been restricted to positive integers; the cases in which they are negative, zero, or fractional are considered later.

MONOMIALS BY MONOMIALS

102. 1. Find the product of $2a^2b$ and $3a^3b^2$.

Since we can change the order of the factors,

$$2a^2b \times 3a^3b^2 \equiv 2 \times 3 \times a^2 \times a^3 \times b \times b^2 \equiv 6a^5b^3.$$

Find the products:

2. -2×-5.
3. $3a \times 4b$.
4. $2x^2y \times 3xy^2$.
5. $3bc \times -mn$.
6. $5x^2y^5 \times 2x^2y^2$.
7. $xy^2z^4 \times x^4y^3z^2$.
8. $a^2b^2c \times abc$.
9. $a^3 \times a^2 \times b^2 \times b^3$.
10. $-a \times -a^2 \times n$.
11. $-a \times (-a)^2$.
12. $-m \times -m^2 \times -m^3$.
13. $-2x^myz \times 3x^3y^5$.
14. $a^mby \times a^2b^ny$.
15. $(-m) \times (-m^2) \times (-m^5)$.
16. $a^m \times a^{2m} \times ab$.
17. $x^my^n \times x^ny^m$.
18. $(5a^mx)(-3a^2x^n)$.
19. $y^{m-n+1} \times y^{n+1}$.
20. $a^{m-1}b^{m-2}c^2 \times a^{m+1}b^2c^3$.

POLYNOMIALS BY MONOMIALS

103. In finding the product of the factors 3 and $(2 + 5)$, we may multiply the sum of 2 and 5 by 3, or multiply 2 by 3 and 5 by 3, and add the products. By the first method we have

$$(2 + 5)3 = 7 \times 3 = 21.$$

By the second, we have

$$(2 + 5)3 = 2 \times 3 + 5 \times 3 = 6 + 15 = 21.$$

This example suggests the following method of multiplying a polynomial by a monomial:

Multiply each term of the polynomial by the monomial and add the products.

104. The principle illustrated by the preceding example and upon which the suggested method is founded is called the **Distributive Law for Multiplication.** We have the following cases:

I. When the monomial is a *positive integer,* it is to be proved that $$(b + c)a \equiv ba + ca.$$

By the definition of multiplication, we have

$$(b + c)a \equiv (b + c) + (b + c) + (b + c) + \cdots \text{ to } a \text{ terms},$$
$$\equiv (b + b + \cdots \text{ to } a \text{ terms}) + (c + c + \cdots \text{ to } a \text{ terms}),$$
(by Commutative and Associative laws)
$$\equiv ba + ca.$$

Similarly in the case of $(b - c)a$, we have

$$(b - c)a \equiv (b + b + \cdots \text{ to } a \text{ terms}) - (c + c + \cdots \text{ to } a \text{ terms}),$$
$$\equiv ba - ca.$$

II. When the monomial is a *negative integer,* it is to be proved that $$(b + c)(-n) \equiv -bn - cn.$$

Since $-n = -(+n)$, it is seen that $-n$ times any number gives the same result as n times the number preceded by the sign $-$. But by I,

$$\because (b + c)n \equiv bn + cn,$$
$$\therefore (b + c)(-n) \equiv (b + c)n \text{ preceded by the sign } -,$$
$$\equiv -(bn + cn),$$
$$\equiv -bn - cn, \text{ removing parentheses.}$$

III. As we have seen, the like operations with arithmetical integers readily suggest the Distributive Law, and this we have found to hold for all integers. If we now take the example

$$\tfrac{3}{4}(4 + 8) = \tfrac{3}{4} \times 4 + \tfrac{3}{4} \times 8 = 9,$$

we find that the same law holds for this particular case when the monomial is a numerical fraction. Hence it is reasonable

to infer that the law holds for any positive or negative fractional number. By reasoning similar to the above it can be proved that it does so hold.

1. Multiply $a - b + 2x$ by $3a$.

$$(a - b + 2x)\,3a \equiv 3a^2 - 3ab + 6ax.$$

Each term of the multiplicand is multiplied by $3a$. The algebraic sum of these products is the required product.

CHECK. If $a = 1$, $b = 2$, and $x = 3$,
then $(1 - 2 + 6)3 = 3 - 6 + 18$.

Perform the multiplications and check results:

2. $x^2(x^2 + y^2 + z^2)$.
3. $3m^2(m - n)$.
4. $5a(2 - 3a)$.
5. $-m(m - n - p)$.
6. $-7x(2x^2 + 5x + 7)$.
7. $ab(a^2b + 3ab - b^2)$.
8. $2xyz(x + y + z)$.
9. $abc(ab + bc - ac)$.
10. $-2ab(3a^2b^2 - a^3b^3)$.
11. $-7x^2(2x^3 - 5x - 7)$.
12. $x^2(x^m - x^{m-2}y + x^{m-3}y^2)$.
13. $-5ab(3a^2 - 2ab + 4b^2)$.
14. $x^n(1 + x^n + x^m)$.
15. $m^2(m^2n - 3mn^2 - m^{a+2})$.
16. Multiply $3x + 2a(x + 2ay + 5z^2)$ by $3a^2y$.
17. Multiply $(x + y)^2 - 3a(x + y) + b^2$ by $2(x + y)$.
18. Multiply $2a + 3b(a^2 + 2ab + b^2)$ by abc.
19. Multiply $5m + 4n(m^2 - 2mn + n^2)$ by $2mn$.
20. Multiply $x^2 - 3x + 6b - 1$ by $-x$; by -2.

POLYNOMIALS BY POLYNOMIALS

105. A polynomial is said to be arranged according to the powers of some letter when the exponents of that letter either increase or decrease from term to term.

Thus, $x^3 + 3x^2y + 3xy^2 + y^3$ is arranged according to the *descending* powers of x and the *ascending* powers of y. What is meant by descending powers?

106. An expression is said to be **homogeneous** in x and y, or in more letters, when all of its terms in those letters are of the same degree.

Thus, $x^2 + 2xy + y^2$ is homogeneous in x and y, since in the second term the sum of the exponents of x and y is 2. Why is $xy^3 + x^2y^2 + x^2y^4$ not homogeneous? The expression $axy^3 + bx^2y^2 + cy^4$ is homogeneous as to x and y, but not as to a, x, and y, or as to x or y alone.

107. An expression is said to be **symmetric** with respect to certain letters when those letters can be interchanged without changing the value of the expression.

Thus, $a^2 + 2ab + b^2$ is symmetric as to a and b, since if they are interchanged it becomes $b^2 + 2ba + a^2$, which has the same value. Is $a^2 - b^2$ symmetric?

108. If two expressions are homogeneous, their product is homogeneous; if they are symmetric as to any letters, their product is symmetric as to those letters. Hence it follows that symmetry and homogeneity furnish useful checks in multiplication.

It is evident that if both factors are homogeneous, one being of the mth degree, the other of the nth degree, then each term in the product will be of the $(m + n)$th degree. If not, it is known at once that there is an error. Why is symmetry also a check?

109. The multiplication of a polynomial by a polynomial involves an application of the Distributive Law in the terms of both multiplier and multiplicand. This law in the general form is expressed in the identity

$$(a + b)(c + d) \equiv ac + bc + ad + bd.$$

It is proved as follows:

If we let $m \equiv (a + b)$, we have, by substitution,

$$\begin{aligned}(a + b)(c + d) &\equiv m(c + d) \equiv mc + md,\\ &\equiv (a + b)c + (a + b)d, \text{ because } m \equiv (a + b),\\ &\equiv ac + bc + ad + bd, \text{ by Art. 104, I.}\end{aligned}$$

From the above is derived the following method of multiplying a polynomial by a polynomial:

Multiply each term of the multiplicand by each term of the multiplier, and add the products (algebraically).

1. Multiply $x^3 + 3x^2y + 3xy^2 + y^3$ by $x + y$.

$$\begin{array}{ll}
x^3 + 3x^2y + 3xy^2 + y^3 & \\
x + y & \\
\hline
x^4 + 3x^3y + 3x^2y^2 + xy^3, & \text{the product by } x. \\
\quad + x^3y + 3x^2y^2 + 3xy^3 + y^4, & \text{the product by } y. \\
\hline
x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4, & \text{the complete product.}
\end{array}$$

CHECK.
$$\begin{array}{rl}
1 + 3 + 3 + 1 = & 8 \\
1 + 1 \quad\quad = & 2 \\
\hline
1 + 4 + 6 + 4 + 1 = & 16
\end{array}$$

1. It is a convenient arrangement to write the multiplier under the multiplicand, and place like terms of the partial products in columns.

2. Observe that in multiplying we take the *product* of the *coefficients* and the *sum* of the *exponents* of the same letters. Is it simpler to multiply from left to right than from right to left?

3. We may check as above by letting $x = 1$ and $y = 1$. Since any power of 1 is 1, the value 1 does not check the exponents. Would the values $x = 2$, $y = 3$?

4. What is the advantage of arranging both factors according to the powers of some letter?

2. Multiply $x^{3n} - x^{2n} + x^n - 1$ by $x^n + 1$.

$$\begin{array}{l}
x^{3n} - x^{2n} + x^n - 1 \\
x^n + 1 \\
\hline
x^{4n} - x^{3n} + x^{2n} - x^n \\
\quad + x^{3n} - x^{2n} + x^n - 1 \\
\hline
x^{4n} \qquad\qquad\qquad - 1
\end{array}$$

Multiply, and check results:

3. $a + b$ by $a + b$.
4. $a - b$ by $a - b$.
5. $a + b$ by $a - b$.
6. $x + 9$ by $x - 9$.
7. $m + 5$ by $m - 2$.
8. $3y + 7z$ by $4z + 5y$.
9. $8a - 6b$ by $6b - 8a$.
10. $x + a$ by $x + b$.

11. $x + m$ by $x - n$.

12. $x^2 + 2xy + y^2$ by $x + y$.

13. $a^2 + ab + bc - b^2$ by $a^2 + b^2$.

14. $r^2 - r + 1$ by $r^2 + r + 1$.

15. $x^3 + x^2y^2 - z^3$ by $x - y$.

16. $m^3 + 3m^2n + mn^2$ by $m - n$.

17. $8am + 6bn$ by $5ab - b$.

18. $b + c$ by $b^2 - c - (a + 1)$.

19. $ax^2 + by^2$ by $ax^2 - by^2$.

20. $ax + b$ by $cx + d$.

21. $a^2 + 2ab + b^2$ by $a^2 - 2ab + b^2$.

22. $a + b + c$ by $a - b - c$.

23. $y^6 - y^4 + y^2 - 1$ by $y^2 + 1$.

24. $a^3 - 3a^2b + 3ab^2 - b^3$ by $a - b$.

25. $x^5 - x^4 + x^3 - x^2 + x - 1$ by $x + 1$.

26. $a^2 - 4a^2b + 2ab^2 - b^3$ by $a^2 - 3ab + b^2$.

27. $5x^3 - 2x^2 - 3x + 4$ by $3x^2 - 2x$.

28. $ax^3 - x^2 + 3x - b$ by $ax^3 + x^2 + 3x + b$.

29. $\frac{1}{2}x - \frac{1}{3}y$ by $\frac{1}{3}x + \frac{1}{2}y$.

30. $x^{2n} - 2x^ny^n + y^{2n}$ by $x^{2n} + 2x^ny^n + y^{2n}$.

31. $16p^2 + 20pq + 25q^2$ by $4p - 5q$.

32. $x^{n+2} + 2x^{n+1} - 3x^n - 1$ by $x + 1$.

33. $\frac{1}{2}x^2 + xy + y^2$ by $\frac{1}{2}x^2 + y^2 - xy$.

34. $(a^n + b^n + c^n + d^n)$ by itself.

35. $x^n + y^n + z^n - 1$ by $x^m + y^m + z^m$.

36. $x^{n+1} + y^{n+1}$ by $x^{n+1} - y^{n+1}$.

37. $x^{n+2} + 2x^{n+1} + 1$ by $x^{n-2} - 4$.

38. $4y^{2n-1} - 5z^{1+2n} - 2$ by $y - z^{2n-1} + 2$.

39. Multiply $a^3 - 3a^2b + 3ab^2 - b^3$ by $a^2 - 2ab + b^2$, checking the product by homogeneity.

40. Multiply $x^3 + y^3 + z^3 + 2xyz$ by $x + y$, checking the product by symmetry.

41. Multiply $3y - 7xy^3 + 5x^3y^2 + 4x^2y^4$ by $x - y$, arranging according to descending powers of x.

Simplify the following expressions:

42. $(x+2)(x-2)-(x+1)(x+3)$.

43. $(x+y)^2-(x+z)^2-(y+z)^2$.

44. $(a+b+c)(a+b-c)(c+a-b)$.

45. $(1+x)^2+(1-x)^2-2(x^2-1)+5$.

46. $(x^2+y^2)^2-(x^2-y^2)^2-(2xy)^2$.

47. $(ax+bx^2+cx^3)(ax+bx^2-cx^3)$.

48. $(a^2+ab+b^2)(a^3-a^2b+b^3)(a-b)$.

49. $(m+n)m-[(m-n)^2-n(n-m)]$.

50. $(a+b)(b+c)-(c+d)(d+a)-(a+c)(b-d)$.

110. Detached Coefficients can be employed to advantage in the multiplication of polynomials whenever the literal part of the product is easily discovered by inspection. This is evidently the case when two polynomial factors involve but one letter, or when they are homogeneous and involve but two letters.

1. Multiply $a^2+2ab+b^2$ by $a+b$; also by a^2+b^2.

(a)		(b)
$1+2+1$	$=4$	$1+2+1$
$1+1$	$=2$	$1+0+1$
$1+2+1$		$1+2+1$
$1+2+1$		$1+2+1$
$1+3+3+1$	$=8$	$1+2+2+2+1$

$\therefore (a+b)(a^2+2ab+b^2) \equiv a^3+3a^2b+3ab^2+b^3$,

and $(a^2+b^2)(a^2+2ab+b^2) \equiv a^4+2a^3b+2a^2b^2+2ab^3+b^4$.

1. In (a) it is apparent that the exponents of a decrease by 1, while those of b increase by 1 in each factor. Does this law hold good in the product? When the coefficients are known, can the product be readily written?

2. If any power is lacking in either factor, a zero coefficient must be used when the work is done by the method of detached coefficients, as in (b). It should be observed that x^3+3x+5, for example, may be written x^3+0x^2+3x+5. Why?

Multiply, using detached coefficients when practicable:

2. $a^2 - ab + b^2$ by $a - b$.
3. $(x - y)(x - y)$.
4. $(x + y)(x - y)$.
5. $(x + y + z)(x + y + z)$.
6. $(x + y)(x + y)(x + y)$.
7. $(a^2 - ab + b^2)(a^2 - ab + b^2)$.
8. $x^2 - xy + y^2$ by $x^2 - 2xy + y^2$.
9. $x^3 + x^2 + x + 2$ by $x - 7$.
10. $a^2b + ab^2 - b^3$ by $a^2 - ab + b^2$.
11. $x^3 - x^2y + xy^2 - y^3$ by $x - y$.
12. $(x + y)(y + z)$ by $x + z$.
13. $a^3 - a^2 - a + 1$ by $a + 1$.
14. $(x - 1)(x - 2)(x - 3)$.
15. $(a^2 + ab + b^2)(a^3 - a^2b + b^3)(a - b)$.

Verify the following identities:

16. $(x^2 + y^2)(m^2 + n^2) - (mx + ny)^2 \equiv (my - nx)^2$.
17. $(x - y)(x^2 + xy + y^2) \equiv (x - y)^3 + 3xy(x - y)$.
18. $(a^2 + ab + b^2)^2 - a^2b^2 \equiv (a^2 + ab)^2 + (ab + b^2)^2$.
19. $a(a - 2b)^3 - b(b - 2a)^3 \equiv (a - b)(a + b)^3$.
20. $(a - b)^3 + (b - c)^3 + (c - a)^3 \equiv 3(a - b)(b - c)(c - a)$.
21. $(x + y + z)(xy + yz + xz) \equiv (x + y)(y + z)(x + z) + xyz$.

POWERS

111. As we have already seen, the product of two or more *equal* factors is called a **Power.** In this limited sense, a^m means the product of m factors, each a; the exponent m therefore is necessarily a positive integer. Hereafter, however, we shall see that the meaning of the word "power" is very much extended.

112. As already stated, a^m means $aaa \cdots$ to m factors. The expression $(a^m)^n$ is taken to mean $a^m \times a^m \times a^m \cdots$ to n factors, each a^m.

Thus, $(a^2)^2$ means $a^2 \times a^2 \equiv a^{2+2} \equiv a^4$.

NOTE. Negative factors should be inclosed in parentheses when there is danger of ambiguity. Thus, $(-a)^2$ means $-a \times -a \equiv +a^2$, while $-a^2$ means $-aa$.

113. In proving the Law of Exponents for multiplication, we established the principle for the product of powers, where the exponents are positive integers. It was shown that

$$a^m \times a^n \equiv a^{m+n}.$$

That is, *the product of the* **m**th *power and the* **n**th *power of* **a** *equals the* $(\boldsymbol{m+n})$**th** *power of* **a**.

114. To prove that $(a^m)^n \equiv a^{mn}$, where m and n are positive integers, we proceed as follows:

$$\begin{aligned}(a^m)^n &\equiv a^m \times a^m \times a^m \cdots \text{ to } n \text{ factors},\\ &\equiv a^{m+m+m \cdots \text{ to } n \text{ terms}}, \qquad \text{(Law of Exp.)}\\ &\equiv a^{m \times n} \equiv a^{mn}.\end{aligned}$$

That is, *the* **n**th *power of the* **m**th *power of* **a** *equals the* **mn**th *power of* **a**.

115. Since $(ab)^m$ means $(ab) \times (ab) \times (ab) \cdots$ to m factors, each (ab), it is readily shown that $(ab)^m \equiv a^m b^m$, where m is a positive integer. Thus we have

$$\begin{aligned}(ab)^m &\equiv (ab) \times (ab) \times (ab) \cdots \text{ to } m \text{ factors},\\ &\equiv (aaa \cdots \text{ to } m \text{ factors}) \times (bbb \cdots \text{ to } m \text{ factors}), \text{ (by Assoc.}\\ &\equiv a^m \times b^m \equiv a^m b^m. \qquad \text{and Com. Laws)}\end{aligned}$$

Similarly, the law can be shown to hold for any number of factors. That is,

The **m**th *power of a product equals the product of the* **m**th *powers of its factors.*

116. Since $^+a \times {}^+a \equiv {}^+a^2$, $^-a \times {}^-a \equiv {}^+a^2$, $^-a \times {}^-a \times {}^-a \equiv {}^-a^3$, etc., the following laws are evident:

I. **All powers of positive numbers are positive.**

II. **Even powers of negative numbers are positive, but odd powers are negative.**

117. Since $(a^2)^2 \equiv a^4$, and $(a^3)^2 \equiv a^6$, it is readily seen that in raising expressions to the 4th power we may conveniently square the expression and then square that result; and in raising to the 6th power we may first cube, and then square the result.

Thus, $(x+y)^4 \equiv (x^2+2xy+y^2)^2$;

and $(x^2+2xy+y^2)^2 \equiv x^4+4x^3y+6x^2y^2+4xy^3+y^4$.

Find the value of each of the following:

1. 3^2.
2. $(-2)^3$.
3. $(ab)^4$.
4. $(a^4)^3$.
5. $(-a)^4$.
6. $(-x)^5$.
7. $(-3)^2$.
8. $(-a^4)^3$.
9. $(ab)^n$.
10. $(2xy^2)^4$.
11. $(abc)^3$.
12. $(-5a^2b^2)^3$.
13. $(a^2)^6$.
14. $(a^m)^n$.
15. $(a^n)^m$.
16. $(-a^mb^n)^m$.
17. $(abc)^n$.
18. $(m+n)^{2+1}$.
19. Prove that $(a^m)^n \equiv (a^n)^m$.

Raise to the indicated power:

20. $(x+2)^2$.
21. $(a+b)^3$.
22. $(x^n-y^n)^2$.
23. $(a-1)^4$.
24. $(x^3-y^3)^3$.
25. $(1-n)^6$.
26. $(\frac{1}{2}a-\frac{1}{2}b)^2$.
27. $(x+y+z)^3$.
28. $(a^2+2ab+b^2)^2$.
29. $(a-2b+c)^2$.
30. $(a^4+a^2b^2+y^4)^2$.
31. $(x^3+x^2-1)^2$.
32. $(a-b-c)^2$.
33. $(a^m+x^n-y^{n+1})^2$.

IMPORTANT PRODUCTS

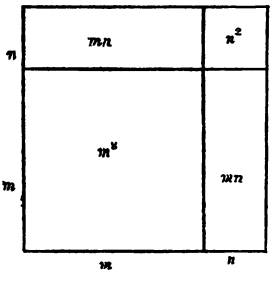
118. The following products are of such importance and occur so frequently that they demand special attention. By reference to certain *type forms* we are enabled to write similar products without the labor of performing the actual multiplications. Those most useful in practice are here considered.

I. **The square of the sum of two numbers.**

By multiplication, we have

$$(x+y)^2 \equiv (x+y)(x+y) \equiv x^2+2xy+y^2.$$

That is, *the square of the sum of two numbers is equal to the square of the first,* plus *twice their product,* plus *the square of the second.*



Thus, $(4+3)^2 = 4^2+2(4\times 3)+3^2 = 49$;

and $(a+2b)^2 \equiv a^2+2(a\times 2b)+4b^2,$

$$\equiv a^2+4ab+4b^2.$$

Illustrate this theorem by solving the problem of finding the number of square rods in a field $m+n$ rods square. Use the figure.

II. **The square of the difference of two numbers.**

By multiplication, we have

$$(x-y)^2 \equiv (x-y)(x-y) \equiv x^2-2xy+y^2.$$

That is, *the square of the difference of two numbers is equal to the square of the first,* minus *twice their product,* plus *the square of the second.*

Thus, $(5-3)^2 = 5^2-2(5\times 3)+3^2 = 4$;

and $(2a-b)^2 \equiv 4a^2-2(2a\times b)+b^2,$

$$\equiv 4a^2-4ab+b^2.$$

III. **The product of the sum and difference of two numbers.**

By multiplication, we have

$$(x+y)(x-y) \equiv x^2-y^2.$$

That is, *the product of the sum and difference of two numbers is equal to the difference of their squares.*

Thus, $(5+3)(5-3) = 25-9 = 16$;

and $(2a+3b)(2a-3b) \equiv 4a^2-9b^2.$

IV. The cube of the sum of two numbers.

By multiplication we have

$$(x+y)^3 \equiv x^3 + 3x^2y + 3xy^2 + y^3. \qquad (A)$$

That is, *the cube of the sum of two numbers is equal to the sum of their cubes,* plus *three times the square of the first into the second,* plus *three times the first into the square of the second.*

V. The cube of the difference of two numbers.

By multiplication, we have

$$(x-y)^3 \equiv x^3 - 3x^2y + 3xy^2 - y^3. \qquad (B)$$

That is, *the cube of the difference of two numbers is equal to the difference of their cubes,* minus *three times the square of the first into the second,* plus *three times the first into the square of the second.*

QUERIES. (1) Do the coefficients in A differ from those in B?
(2) Are A and B homogeneous expressions?
(3) In B, do + and − alternate?

By the aid of these theorems write the following products:

1. $(a+b)^2$.
2. $(a-b)^2$.
3. $(x+2)^2$.
4. $(2a+b)^2$.
5. $(x^2-1)^2$.
6. $(c+3)^2$.
7. $(x+2a)^2$.
8. $(x^2-y)^2$.
9. $(ab+3)^3$.
10. $(a^2b^2-c^2)^2$.
11. $(r-3)^3$.
12. $(x^3+y^2)^2$.
13. $(2x+y)^3$.
14. $(2p+3q)^2$.
15. $(2p-3q)^2$.
16. $(3a-1)^2$.
17. $(x^2y+xy^2)^2$.
18. $(3xy-2z)^2$.
19. $(2x^m-7)^2$.
20. $(y^n-4)^3$.
21. $(x^n+y^n)^2$.
22. $(y+2)^3$.
23. $(m-n)^3$.
24. $(x+4)(x+4)$.
25. $(c+4d)(c-4d)$.
26. $(x^2+7)(x^2-7)$.

27. $(m+n)(m-n)$.
28. $(x^n+1)(x^n-1)$.
29. $(2a+5b)(2a-5b)$.
30. $(a^2b-1)(a^2b+1)$.
31. $(x^3+y^3)(y^3-x^3)$.
32. $(2ab+c)(2ab-c)$.
33. $(x^{m+1}-y^{n-2})^2$.
34. $(b^2+a^2)(a^2-b^2)$.
35. $[(x^2+1)+x][(x^2+1)-x]$.
36. $[x-(y-z)][x+(y-z)]$.
37. $(1+x+y)(1+x-y)$.
38. What two equal factors produce $a^2+2ax+x^2$?
39. What two equal factors give the product $b^2-2ab+a^2$?
40. What two factors produce m^2-n^2? $4x^2-16y^2$?

THE BINOMIAL THEOREM

119. A very useful law for raising binomials to various powers is known as the **Binomial Theorem**. Examine carefully the following powers, and verify the results by actual multiplication:

$(a+b)^2 \equiv a^2+2ab+b^2$.

$(a-b)^2 \equiv a^2-2ab+b^2$.

$(a+b)^3 \equiv a^3+3a^2b+3ab^2+b^3$.

$(a-b)^3 \equiv a^3-3a^2b+3ab^2-b^3$.

$(a+b)^4 \equiv a^4+4a^3b+6a^2b^2+4ab^3+b^4$.

$(a-b)^4 \equiv a^4-4a^3b+6a^2b^2-4ab^3+b^4$.

$(a+b)^5 \equiv a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5$.

$(a+b)^6 \equiv$ (Write this.)

$$(a+b)^n \equiv a^n+na^{n-1}b+\frac{n(n-1)}{2}a^{n-2}b^2+\frac{n(n-1)(n-2)}{2\times 3}a^{n-3}b^3+\cdots,$$

n being a positive integer.

How does the number of terms in each result compare with the degree of the binomial?

How does the highest exponent of a in each case compare with the exponent of the binomial? How do the exponents of a decrease from term to term? How do the successive exponents of b change?

What is the first coefficient in each case? Is the second coefficient the same as the exponent of the binomial?

Note the 4th power. The second coefficient multiplied by the exponent of $a = 4 \times 3$, or 12. This divided by one more than the exponent of $b = 12 \div 2$, or 6. Hence 6 is the third coefficient. To get the next coefficient we multiply 6 by 2, and divide by 3. Find the third coefficient in the case of the 5th power. The fourth coefficient.

Are the coefficients of terms equally distant from the beginning and the end equal? May we then write down, without calculation, the coefficients after the middle term? Is each product homogeneous in a and b?

Note how the signs alternate when the second term of the binomial is negative.

From the preceding examples and exercises, the following laws are evident:

I. **The number of terms is 1 more than the binomial exponent.**

II. **The exponent of a in the first term, as well as that of b in the last term, is equal to the binomial exponent. The exponents of a decrease by 1 from term to term, while those of b increase by 1.**

III. **The first coefficient is 1, the second is equal to the binomial exponent; and the coefficient of any succeeding term is found by multiplying the coefficient of the preceding term by the exponent of a in that term and dividing by 1 more than the exponent of b.**

IV. **The signs of the terms are alternately + and − when the last term of the binomial is negative; otherwise they are all positive.**

NOTE. The formal proof of the Binomial Theorem is deferred. The early use of the theorem, however, is of great importance, since by it we are enabled to raise a binomial to any power without the trouble of actual multiplication.

1. Raise $(2x + 3y)$ to the 3d power.

$$(2x + 3y)^3 \equiv (2x)^3 + 3(2x)^2(3y) + 3(2x)(3y)^2 + (3y)^3,$$
$$\equiv 8x^3 + 36x^2y + 54xy^2 + 27y^3.$$

CHECK. $(5)^3 = 8 + 36 + 54 + 27 = 125.$

2. Square $x + y - z$.

$$(x + y - z)^2 \equiv [(x + y) - z]^2,$$
$$\equiv (x + y)^2 - 2(x + y)(z) + z^2,$$
$$\equiv x^2 + 2xy + y^2 - 2xz - 2yz + z^2.$$

By properly grouping the terms, expressions of three or more terms can frequently be treated as binomials.

Raise to the indicated powers:

3. $(x + y)^5$.
4. $(x - y)^5$.
5. $(1 - 2x)^6$.
6. $\left(x + \frac{1}{y}\right)^3$.
7. $(a^2 + ax)^4$.
8. $(2a - 3b)^3$.
9. $(x - y)^7$.
10. $(1 + a)^3$.
11. $(a + 3x)^3$.
12. $\left(x + \frac{1}{x}\right)^4$.
13. $(x^m + y^m)^2$.
14. $(5a + 2b^4)^2$.
15. $(a - b^3)^4$.
16. $(ab - 2b^2)^2$.
17. $(\frac{1}{2}x - y)^5$.
18. $\left(1 - \frac{1}{x^2}\right)^4$.
19. $(c - \frac{1}{2}c)^3$.
20. $(x^2 - 3xy)^5$.
21. $(x - y + z)^2$.
22. $(a + b + c)^3$.
23. $(m + n - a)^4$.
24. $(x - y - z + w)^2$.
25. $(x^2 - y^2 + z^2)^3$.
26. $(ab + 3 + c)^2$.
27. $(ab + bc + ac)^4$.
28. $[(a + b + c)^2 + d]^2$.

EXERCISES

120. 1. Explain the extended meaning of the word "times" as used in the definition of multiplication. Is the *many-fold* idea the same whether the multiplier is positive or negative?

2. What is the Distributive Law of Multiplication? Illustrate.

3. In multiplication, why do you arrange the terms according to the power of some letter?

4. Make a problem to illustrate the advantage of using detached coefficients in multiplication.

5. What is the etymological meaning of "commutative"? Of "homogeneous"? Of "symmetric"?

6. In raising $(a+b)$ to the 6th power, what will be the coefficient of a^4b^2? Of a^3b^3?

7. Find the sum of the coefficients in $(x+1)^5$.

8. Multiply $2a^2+3ab+4b^2$ by $a+b$, checking the product by homogeneity.

9. Multiply $x^4-x^3y+x^2y^2-xy^3+y^4$ by $x+y$, checking the product (1) by symmetry, (2) by homogeneity.

10. Find the product of $(a-b)(c-d)(e-f)$. Does each term of the product contain one letter from each factor? Is it homogeneous?

11. How many square feet in a room $a+b$ feet long and $a-b$ feet wide? How many square yards, if $a=15$, $b=12$?

12. Find the volume of a cube whose edge is $m-n$ inches.

13. The number 25 may be written $20+5$. Write the number containing a ones and b tens. Square $10b+a$.

14. What is the value of $a[a^2-2b-2c(a-b)^2]$ when $a=5$, $b=3$, and $c=2$?

15. Express in words the algebraic statement,

$$(a+b)^3 \equiv a^3+b^3+3ab(a+b).$$

16. The cube of the difference of two numbers is equal to the difference of the cubes diminished by three times the product of the numbers multiplied by their difference. Express this algebraically.

DIVISION

121. Division is the inverse of multiplication. In the latter, multiplicand and multiplier are given, while the product is required. In the former, the product (*dividend*) and the multiplier (*divisor*) are given, while the multiplicand (*quotient*) is required.

NOTE. This definition is made to conform strictly to inverse of multiplication as defined in Art. 92. Since the Commutative Law applies to every case of multiplication, we may define division as the operation of finding the other factor when the product and one factor are given.

122. The expression $0 \times a$ has already been shown to be 0; hence by the definition of division, $0 \div a = 0$. But when the product and the given factor are each 0, the other factor cannot be found, since 0 is the product of *any* (finite) number and 0.

123. Law of Signs.

Since division is the inverse of multiplication, the Law of Signs in the former may be derived from that in the latter, as follows:

Since $+5 \times +3 = +15$, $\therefore +15 \div +3 = +5$.

Since $-5 \times +3 = -15$, $\therefore -15 \div +3 = -5$.

Since $+5 \times -3 = -15$, $\therefore -15 \div -3 = +5$.

Since $-5 \times -3 = +15$, $\therefore +15 \div -3 = -5$.

Or, in general,

$+b \times +a \equiv +(ba)$, $\therefore +(ba) \div +a \equiv +b$.

$-b \times +a \equiv -(ba)$, $\therefore -(ba) \div +a \equiv -b$.

$+b \times -a \equiv -(ba)$, $\therefore -(ba) \div -a \equiv +b$.

$-b \times -a \equiv +(ba)$, $\therefore +(ba) \div -a \equiv -b$.

That is, **like signs of dividend and divisor give a positive quotient; unlike signs, a negative quotient.**

124. Law of Exponents.

It has already been shown that

$$a^5 \times a^2 \equiv a^{5+2} \equiv a^7, \quad \therefore\ a^7 \div a^2 \equiv a^5;$$

and since $a^{5-2} \times a^2 \equiv a^{(5-2)+2} \equiv a^5, \quad \therefore\ a^5 \div a^2 \equiv a^{5-2} \equiv a^3.$

Or, in general, when m and n are positive integers,

$$a^m \times a^n \equiv a^{m+n}, \quad \therefore\ a^{m+n} \div a^n \equiv a^m,$$

$$a^{m-n} \times a^n \equiv a^m, \quad \therefore\ a^m \div a^n \equiv a^{m-n}, \text{ if } m > n.$$

That is, **the exponent of a letter in the quotient is equal to its exponent in the dividend diminished by its exponent in the divisor.**

125. Since any number divided by itself is equal to unity,

$$a^m \div a^m \equiv 1, \text{ just as } 5^2 \div 5^2 = 1.$$

By preceding article,

$$a^m \div a^m \equiv a^{m-m} \equiv a^0, \quad \therefore\ a^0 \equiv 1. \qquad \text{(Axiom 1)}$$

Divide the following:

1. $6\,ab$ by 2.
2. $9\,bx$ by $3\,b$.
3. $16\,c$ by $-4\,c$.
4. $-14\,y$ by $2\,y$.
5. $-12\,mb$ by $-3\,m$.
6. a^2b by $-ab$.
7. $-xy^2$ by $-xy$.
8. $-a^2b^2c$ by abc.
9. $20\,m^2by$ by $-4\,m$.
10. $16\,a^2x^2y$ by $8\,axy$.
11. $-6\,x^7y^{11}$ by $6\,x^3y^6$.
12. $-49\,p^2qr$ by $-7\,p^2r$.
13. $x^7 \div x^4$.
14. $a^n \div a^n$.
15. $y^{n+1} \div y^n$.
16. $x^{m+3} \div x^{m-3}$.
17. $a^{2-n} \div a^{n-2}$.
18. $x^{m+2} \div x^{n+1}$.
19. $b^{2n} \div b^{n-1}$.
20. $a^{0+n} \div a^n$.
21. $y^{m+n} \div y^{m-n}$.
22. $-6\,a^{m+n} \div -2\,a^n$.
23. $-27\,a^{m-1} \div -9\,a^{m-3}$.
24. $16\,x^{2m}y^{n+1} \div 4\,x^my^{n-1}$.

25. $-25\,a^2(x-y)$ by $-5\,a(x-y)$.

26. $10\,x^2(x+1)^2$ by $-2\,x(x+1)$.

27. $35(a+b)^3$ by $7(a+b)x^2$.

DIVISION OF A POLYNOMIAL BY A MONOMIAL

126. In finding the quotient of $(6+4)\div 2$, the result is the same whether we divide the sum of 6 and 4 by 2, or divide 6 by 2 and 4 by 2, and add the quotients. (Cf. Arts. 103 and 104.)

By the first method,

$$(6+4)\div 2 = 10\div 2 = 5.$$

By the second,

$$(6+4)\div 2 = 6\div 2 + 4\div 2 = 3+2 = 5.$$

For, in general, since

$$(a+b-c)\times m \equiv ma+mb-mc,$$

$$(ma+mb-mc)\div m \equiv a+b-c, \text{ by def. of division.}$$

These examples indicate the following method of dividing a polynomial by a monomial:

Divide each term of the polynomial by the monomial, and add the quotients (algebraically).

1. Divide $4\,a^2x - 2\,ax^2$ by $2\,ax$.

$$(4\,a^2x - 2\,ax^2)\div 2\,ax \equiv 2\,a - x.$$

CHECK. If $a=2$, $x=3$, then $(48-36)\div 12 = 4-3 = 1$.

Divide the following, checking results:

2. $mx+nx$ by x.

3. a^2b-ab^2 by ab.

4. a^3b-ab^3-ab by $-ab$.

5. $-x+y+z-w$ by -1.

6. $4\,a^2x - 6\,aby^2 + 2\,ac^2$ by $2\,a$.

7. $4\,b^3c - 8\,b^2c^2 + 4\,bc^3$ by $2\,cb$.

8. $8\,x^3 - 4\,x^2y - 6\,xy^2$ by $-2\,x$.

9. $m^2 - m^3n - m^5$ by $-m^2$.

10. $x^2 + x - (2\,x - x^2)$ by x.

11. $4\,a^4b^4 - 8\,a^3b^3 + 6\,ab^3$ by $-2\,ab$.

12. $-20\,a^2b^3c^4+25\,a^4b^3c^2$ by $5\,a^2b^2c^2$.

13. $-x-x^2y-x^3z-x^4y-x^5w$ by $-x$.

14. $-20\,x^{12}y^9-25\,x^9y^{12}-15\,x^6y^6-10\,x^3y^3$ by $-5\,x^3y^3$.

15. $34\,x^3y^3z^4-17\,x^4y^3z^3+17\,x^5y^4z^3$ by $-17\,x^2y^2z^3$.

16. $a^m-a^{m-1}+a^{m-2}-a^{m-3}$ by a^2.

17. $x^{n+3}-2\,x^{n+2}+3\,x^{n+1}+x^n$ by x^{n-1}.

18. $15(a+b)^2+10(a+b)$ by $5(a+b)$.

19. $10(a-1)^2-6\,a(a-1)$ by $(a-1)$.

20. If n is an odd number, what is $n+1$? What kind of numbers is exactly divisible by 2?

21. If n is an even number, what is $n+1$? $n-1$?

22. If m and n are both even numbers, is their sum an even number? Prove it.

23. If m and n are both odd numbers, prove that their sum is even.

24. If m is even and n odd, prove that $m+n$ is odd; also that $m-n$ is odd.

25. When m is even, what is $m+a^0$? $m\times a^0$?

26. When m is even and n odd, are the following odd or even? $(m+n)+(m-n)$; $(m^2-n^2)\div(m-n)$; $(m^2-n^2)+a^0$.

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

127. In multiplication we have seen that, if the factors are arranged according to the descending (or ascending) powers of some letter, the first term of the product is the product of the first terms of the factors. Hence, since division is the inverse process, it follows that if dividend and divisor are arranged in the *same order* with respect to some letter, then the first term of the quotient is the quotient of the first terms of dividend and divisor.

Thus, in dividing $x^3-3\,x^2y+3\,xy^2-y^3$ by $x^2-2\,xy+y^2$, the first term of the quotient is $x^3\div x^2$, or x.

128. Since the *whole dividend* is the product of the divisor by the quotient, it is evident that if the product of the divisor by the first term of the quotient is subtracted from the dividend, the remainder (*partial* dividend) is the product of the divisor by the other terms of the quotient.

Thus, in dividing $x^3 - 3x^2y + 3xy^2 - y^3$ by $x^2 - 2xy + y^2$, if we multiply the divisor by the first term of the quotient, which is x, and subtract the product from the dividend, we have the remainder (partial dividend), $-x^2y + 2xy^2 - y^3$, which is the product of the divisor, $x^2 - 2xy + y^2$, by the other term of the quotient $(-y)$.

129. The method of procedure in this form of division is similar to the "long division" of arithmetic with which we are familiar, and will be best understood by comparing the process in multiplication with the corresponding process in division, as given below.

(a)

$$
\begin{array}{ll}
x^2 - 2xy + y^2 & = \text{one factor (divisor below).} \\
x - y & = \text{other factor (quotient).} \\
\hline
x^3 - 2x^2y + xy^2 & = \text{first partial product.} \\
\quad - x^2y + 2xy^2 - y^3 & = \text{second partial product.} \\
\hline
x^3 - 3x^2y + 3xy^2 - y^3 & = \text{product (dividend below).}
\end{array}
$$

(b)

$$
\begin{array}{lll}
\text{Divisor (factor)} & x - y & = \text{quotient (required factor).} \\
x^2 - 2xy + y^2 & x^3 - 3x^2y + 3xy^2 - y^3 & = \text{dividend (product).} \\
 & x^3 - 2x^2y + xy^2 & \\
 & \quad - x^2y + 2xy^2 - y^3 & \\
 & \quad - x^2y + 2xy^2 - y^3 &
\end{array}
$$

The first term of the product in (a) is $x^2 \times x$, or x^3; hence in (b) the first term of the quotient is $x^3 \div x^2$, or x.

Subtracting x times the divisor from the dividend, we have a remainder which is the product of the divisor by the rest of the quotient.

The first term of this remainder is the first term of the second partial product in (a), and is $x^2 \times (-y)$, or $-x^2y$; hence the second term of the quotient is $-x^2y \div x^2$, or $-y$.

Subtracting $-y$ times the divisor from the remainder, we have no second remainder, and the division is exact.

In practice, the work may be conveniently arranged as in (*c*). The use of *detached coefficients*, as in (*d*), often saves much time and labor. In what cases is this method practicable?

(*c*)

$$\begin{array}{l|l} x^3 - 3x^2y + 3xy^2 - y^3 & x^2 - 2xy + y^2 \\ \underline{x^3 - 2x^2y + \quad xy^2} & \underline{x - y} \\ \quad - x^2y + 2xy^2 - y^3 & \\ \quad \underline{- x^2y + 2xy^2 - y^3} & \end{array}$$

(*d*)

$$\begin{array}{l|l} 1 - 3 + 3 - 1 & 1 - 2 + 1 \\ \underline{1 - 2 + 1} & \underline{1 - 1} \\ \quad -1 + 2 - 1 & \\ \quad \underline{-1 + 2 - 1} & \end{array}$$

130. The above example illustrates the following method of dividing a polynomial by a polynomial:

1. *Arrange dividend and divisor according to the descending (or ascending) powers of some common letter.*

2. *Divide the first term of the dividend by the first term of the divisor, and write the result as the first term of the quotient.*

3. *Multiply the divisor by the first term of the quotient, and subtract the product from the dividend.*

4. *If there is a remainder, treat it as a new dividend and proceed as before.*

131. If the division is not exact, the degree of the last partial dividend will be less than that of the divisor, and this partial dividend is called the remainder.

In general, if $D \equiv$ dividend, $d \equiv$ divisor, $q \equiv$ quotient, and $r \equiv$ remainder, then

$$D - r \equiv q \times d.$$

That is, if the remainder is subtracted from the dividend, the result is the product of the quotient and the divisor. Hence, in checking the work by multiplication we should first subtract the remainder.

1. Divide $x^3 - 4x^2 + x^4 + 5x - 3$ by $2x + x^2 - 3$.

$$\begin{array}{l|l} x^4 + \quad x^3 - 4x^2 + 5x - 3 & x^2 + 2x - 3 \\ \underline{x^4 + 2x^3 - 3x^2} & \underline{x^2 - \quad x + 1} \\ \quad - \quad x^3 - \quad x^2 + 5x & \\ \quad \underline{- \quad x^3 - 2x^2 + 3x} & \\ \qquad\qquad x^2 + 2x - 3 & \\ \qquad\qquad \underline{x^2 + 2x - 3} & \end{array}$$

Observe (1) that we first arrange according to the descending powers of x; (2) that the work is shortened by bringing down terms only as needed.

2. Divide $a^3 + b^3 - c^3 + 3abc$ by $a + b - c$.

$$
\begin{array}{l|l}
a^3 + 3abc + b^3 - c^3 & a + b - c \\
\underline{a^3 + a^2b - a^2c} & \overline{a^2 - ab + ac + b^2 + bc + c^2} \\
\quad - a^2b + a^2c + 3abc & \\
\quad \underline{- a^2b \quad - ab^2 + \quad abc} & \\
\qquad a^2c + ab^2 + 2abc & \\
\qquad \underline{a^2c \quad + \quad abc - ac^2} & \\
\qquad\quad ab^2 + \quad abc + ac^2 + b^3 & \\
\qquad\quad \underline{ab^2 \qquad\qquad + b^3 - b^2c} & \\
\qquad\qquad\quad abc + ac^2 \quad + b^2c & \\
\qquad\qquad\quad \underline{abc \qquad\quad + b^2c - bc^2} & \\
\qquad\qquad\qquad\quad ac^2 \qquad + bc^2 - c^3 & \\
\qquad\qquad\qquad\quad \underline{ac^2 \qquad\quad + bc^2 - c^3} &
\end{array}
$$

Observe that the partial dividends are arranged according to descending powers of a, b being given precedence over c for uniformity.

3. Divide $x^3 - 1$ by $x^2 + x + 1$, detaching the coefficients.

$$
\begin{array}{l|l}
1 + 0 + 0 - 1 & 1 + 1 + 1 \\
\underline{1 + 1 + 1} & \overline{1 - 1} \\
\quad - 1 - 1 - 1 & \\
\quad \underline{- 1 - 1 - 1} &
\end{array}
$$

The required quotient is $x - 1$. The missing powers are indicated by zeros.

CHECK. Let $x = 2$, then

$$(4 + 2 + 1)(2 - 1) = 8 + 0 + 0 - 1, \text{ or } 7 \times 1 = 7.$$

In checking, such arbitrary values should be substituted as shall not make the divisor zero.

4. Divide $x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc$ by $x + c$.

$$
\begin{array}{l|l}
x^3 + (a + b + c)x^2 + (ab + ac + bc)x + abc & x + c \\
\underline{x^3 + (\quad + c)x^2} & \overline{x^2 + (a + b)x + ab} \\
\quad (a + b \quad)x^2 + (ab + ac + bc)x & \\
\quad \underline{(a + b \quad)x^2 + (\quad ac + bc)x} & \\
\qquad\qquad abx \qquad\quad + abc & \\
\qquad\qquad \underline{abx \qquad\quad + abc} &
\end{array}
$$

The operation may be shortened in some cases by the use of parentheses.

Divide the following, checking the results:

5. $a^2+2ab+b^2$ by $a+b$.
6. y^2+3y+2 by $y+1$.
7. $m^2-2mn+n^2$ by $m-n$.
8. a^2-a-90 by $a+9$.
9. $15b^2-8bc-12c^2$ by $3b+2c$.
10. $x^3+3x^2y+3xy^2+y^3$ by $x+y$.
11. $y^2-10yz-24z^2$ by $y+2z$.
12. $16x^2-24xy+9y^2$ by $4x-3y$.
13. $x^3+y^3+3xy-1$ by $x+y-1$.
14. $2x^2-x-6$ by $2x+3$.
15. $5x^3-x+1-3x^4$ by $1+x-x^2$.
16. $x^2-y^2-2yz-z^2$ by $x+y+z$.
17. $a^3-a^2b+2b^3$ by $a+b$.
18. $2.21m^2-1.8mn-1.61n^2$ by $1.3m+.7n$.
19. x^3-y^3 by x^2+xy+y^2.
20. a^5-b^5 by $a-b$.
21. $m^2-mx-m+x$ by $m-1$.
22. a^5+b^5 by $a+b$.
23. x^3+3x^2+3x+1 by x^2+2x+1.
24. $x^5-4x^4+3x^3+3x^2-3x+2$ by x^3-x-2.
25. $x^4-x^3y+2x^2y^2-xy^3+y^4$ by x^2-xy+y^2.
26. $x^5-26x^3+6x^2+5x-1$ by x^3-5x^2+1.
27. $2p^3-8p+p^4+12-7p^2$ by p^2+2-3p.
28. $6x^5-13x^4+4x^3+3x^2$ by $3x^3-2x^2-x$.
29. $a^5-a^4b+a^3b^2-a^3-b^3$ by a^3-a-b.
30. $1+x$ by $1+x^2$ to five terms.
31. a^5-1 by $a+1$ to five terms.
32. $2x^6+1-3x^4$ by $1+2x+x^2$.
33. $a^{2n}+11a^n+3$ by a^n+6.
34. x^8-y^8 by $x^3+x^2y+xy^2+y^3$.
35. 1 by $1+x$ to 4 terms.
36. 1 by $1-x$ to 5 terms.
37. $x^2+(m+n)x+mn$ by $x+m$.

38. $(mn+pq)^2-1$ by $(mn+pq)+1$.

39. $m^{10}+m^5n^5+n^{10}$ by m^2+mn+n^2.

40. $1+2y^5+y^6+2y^7$ by $1+y+y^2$.

41. a^5+b^5 by $a^4-a^3b+a^2b^2-ab^3+b^4$.

Divide the following, using detached coefficients:

42. a^3+b^3 by $a+b$.

43. a^6+b^6 by a^2+b^2.

44. x^6-6x+5 by x^2-2x+1.

45. x^4-2x^2+8x-3 by x^2+2x+1.

46. $x^4+x^3+x^2-9x+6$ by x^2-2x+1.

47. $1+n+n^2+n^3+n^4$ by $1-n$.

48. $2a^4+7a^3-27a^2-8a+16$ by a^2+5a+2.

49. $1+2x^2-7x^4-16x^6$ by $1+2x+3x^2+4x^3$.

50. $a^3+3a^2b+3ab^2+b^3+c^3$ by $a+b+c$.

51. $x^8-3x^7-5x^5+2x^4+5x^3+4x^2+2$ by x^3+2x-1.

52. $1+2z^2-7z^4-16z^6$ by $1-2z+3z^2-4z^3$.

53. $a+a^4-a^3+a^2+1$ by a^2+a+1.

54. $15x^4+16x^3-30x^2-13x+20$ by $5x^3-3x^2-5x+4$.

55. $8x^7y+6x^5y^3-6x^3y^5-8xy^7$ by $4x^4y-6x^3y^2+6x^2y^3-4xy^4$.

IMPORTANT QUOTIENTS

132. In multiplication we considered certain type products that are so frequently used as to deserve special mention. For the same reason the following quotients are also very important:

I. **Difference of two squares.**

By actual division we obtain

$$(a^2-b^2)\div(a+b)\equiv a-b;$$
$$(a^2-b^2)\div(a-b)\equiv a+b.$$

That is, *when the dividend is the difference of the squares of two numbers, and the divisor is the sum of those numbers, the quotient is the difference of those numbers.*

State the principle for the second identity.

II. Sum and difference of two cubes.

By actual division we find that

$$(a^3 + b^3) \div (a + b) \equiv a^2 - ab + b^2;$$
$$(a^3 - b^3) \div (a - b) \equiv a^2 + ab + b^2.$$

That is, *when the dividend is the sum of the cubes of two numbers, and the divisor is the sum of those numbers, the quotient is the square of the first, minus their product, plus the square of the second.*

State the principle for the second identity.

III. Sum and difference of like powers.

By actual division we find that

$(a^4 - b^4) \div (a + b) \equiv a^3 - a^2b + ab^2 - b^3;$

$(a^4 - b^4) \div (a - b) \equiv a^3 + a^2b + ab^2 + b^3;$

$(a^4 + b^4)$ is not divisible by either $a + b$ or $a - b$;

$(a^5 + b^5) \div (a + b) \equiv a^4 - a^3b + a^2b^2 - ab^3 + b^4;$

$(a^5 + b^5)$ is not divisible by $a - b$.

From a consideration of the above identities, and those of I and II, we infer the following principles:

(1) *The* **difference** *of the* **like** *powers of two numbers is divisible by the* **difference** *of the numbers.*

In general, when n is a positive integer,

$$(a^n - b^n) \div (a - b) \equiv a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + ab^{n-2} + b^{n-1};$$

in which

(*a*) *the number of terms in the second member is* n;

(*b*) *the exponent of* a *in the first term is* 1 *less than its exponent in the dividend, and decreases by* 1 *from term to term;*

(*c*) *the exponent of* b *is* 1 *in the second term, and increases by* 1 *from term to term;*

(*d*) *the coefficient of each term is* 1;

(*e*) *the signs of the terms are all* +.

(2) *The* **difference** *of the like* **even** *powers of two numbers is divisible by the* **sum** *of the numbers.*

The **sum** *of the like* **odd** *powers of two numbers is divisible by the* **sum** *of the numbers.*

In general, when n is *even*,

$$(a^n - b^n) \div (a + b) \equiv a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + ab^{n-2} - b^{n-1};$$

and when n is *odd*,

$$(a^n + b^n) \div (a + b) \equiv a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - ab^{n-2} + b^{n-1}.$$

Observe that in these quotients the laws of exponents and the number of terms are the same as in (1), but *the signs of the terms are alternately* $+$ *and* $-$.

Are the quotients homogeneous?

1. Write the quotient of $(16\,a^4b^8 - 81\,x^8y^{12}) \div (2\,ab^2 + 3\,x^2y^3)$.

$$\frac{16\,a^4b^8 - 81\,x^8y^{12}}{2\,ab^2 + 3\,x^2y^3} \equiv \frac{(2\,ab^2)^4 - (3\,x^2y^3)^4}{2\,ab^2 + 3\,x^2y^3}$$

$$\equiv (2\,ab^2)^3 - (2\,ab^2)^2(3\,x^2y^3) + (2\,ab^2)(3\,x^2y^3)^2 - (3\,x^2y^3)^3$$

$$\equiv 8\,a^3b^6 - 12\,a^2b^4x^2y^3 + 18\,ab^2x^4y^6 - 27\,x^6y^9.$$

Write the quotients in the following divisions:

2. $\dfrac{x^2 - 25}{x - 5}$.

3. $\dfrac{x^2 - 16}{x + 4}$.

4. $\dfrac{1 - x^4}{1 - x}$.

5. $\dfrac{1 - 64\,a^6c^9}{1 - 4\,a^2c^3}$.

6. $\dfrac{x^4y^8 - 1}{xy^2 + 1}$.

7. $\dfrac{x^6 - y^6}{x - y}$.

8. $\dfrac{x^6 + y^6}{x^2 + y^2}$.

9. $\dfrac{x^8 - y^8}{x^4 + y^4}$.

10. $\dfrac{x^{10} - y^{10}}{x - y}$.

11. $\dfrac{x^{2n} - y^{2n}}{x^n - y^n}$.

12. $\dfrac{x^3 + 27}{x + 3}$.

13. $\dfrac{1 - x^3}{1 - x}$.

14. $\dfrac{1 - 64\,m^2n^2}{1 + 8\,mn}$.

15. $\dfrac{64\,x^{12} - 27\,y^9}{4\,x^4 - 3\,y^3}$.

16. $\dfrac{a^5 + 32}{a + 2}$.

17. $\dfrac{x^{2n} - 1}{x^n - 1}$.

18. $\dfrac{x^2y^2 - \frac{1}{9}}{xy + \frac{1}{3}}$.

19. $\dfrac{(x + y)^2 - 1}{(x + y) + 1}$.

20. $\dfrac{(x + y)^3 - 8}{(x + y) - 2}$.

21. $\dfrac{(a^4 - 2\,a^2b^2 + b^4) - 16}{(a + b)(a - b) - 4}$.

THE REMAINDER THEOREM

133. An expression is **Rational** when it contains no indicated root. It is said to be rational with respect to any letter involved when that letter is free from fractional exponents or other root symbols.

Thus, $x^2 + x + 2$ is a rational expression; $a + \sqrt{b}$ is rational with respect to a, but *irrational* with respect to b.

134. A rational expression is **Integral** when none of the letters involved occurs in any denominator. It is said to be integral with respect to any letter when that letter does not occur in any denominator.

Thus, $2 - \frac{1}{3}x$ is an integral expression, because it has no *algebraic* fraction, the integrality having reference to the *literal* part only; but $x - \frac{3}{x}$ is *fractional* with respect to x, having x in the denominator.

135. THEOREM. **If any rational integral expression involving x is divided by $x - a$, the remainder will be the same as the dividend with a substituted for x.**

That this proposition may be more easily understood, consider first the following example:

$$\begin{array}{l|l} x^2 + mx + n & x - a \\ \cline{2-2} x^2 - ax & x + m + a \\ \cline{1-1} \quad mx + ax + n & \\ \quad mx - ma & \\ \cline{1-1} \quad\quad ax + ma + n & \\ \quad\quad ax - a^2 & \\ \cline{1-1} \quad\quad\quad a^2 + ma + n = \text{remainder.} & \end{array}$$

It is seen that the remainder is the same as the dividend with a substituted for x.

That this is true in all cases may be proved as follows:

Let E denote any rational integral expression involving x, and divide E by $x - a$ until the remainder, if there be a remainder, does not

contain x; and let q be the quotient and r the remainder. Then, since the dividend equals the product of the quotient and the divisor plus the remainder, we have $E \equiv q(x-a)+r$.

Since this is an *identity*, it is true whatever the value of x. Hence, after substituting a for x, E now involving a in place of x, we have

$$E \equiv q(a-a)+r \equiv 0+r \equiv r,$$

where r is not changed when a replaces x, since it does not contain x.

This is known as the **Remainder Theorem.** By its aid we may find the remainder arising from the division, for example, of $x^2-7x+11$ by $x-2$ by merely substituting 2 for x in the dividend. Thus, $4-7\times 2+11=1$, the remainder.

136. THE FACTOR LAW. **If any integral expression involving x becomes zero when a is substituted for x, the expression is exactly divisible by $x-a$.**

By the preceding proof the remainder equals the dividend with a substituted for x, and this being 0 by the supposition, the division is exact. Hence it is known at once that x^3-2x^2+1, for example, is exactly divisible by $x-1$, since $1-2+1=0$.

137. The principles already stated in Art. 132, III, can now be established.

When n is a positive integer:

(*a*) x^n+y^n is divisible by $x+y$ only when n is odd.

Since $x+a \equiv x-(-a)$, $-a$ evidently replaces a in the above proof; hence, in the present case, putting $-y$ for x, x^n+y^n becomes $(-y)^n+y^n$, which is equal to 0 when n is odd, but not when n is even.

Does x^7+y^7 become 0 when $-y$ is put for x? Is it exactly divisible by $x+y$?

(*b*) x^n+y^n is never divisible by $x-y$.

For, putting y for x, x^n+y^n becomes y^n+y^n, or $2y^n$, which is not 0.

(*c*) x^n-y^n is divisible by $x+y$ only when n is even, and not otherwise.

Put $-y$ for x, and give the proof.

(*d*) x^n-y^n is always divisible by $x-y$.

Give the proof.

Find the remainders in the following divisions:

1. $x^3-6x^2+11x+2$ by $x-2$.
2. x^3+2x^2-5x-6 by $x-4$.
3. x^3+2x^2-4x+9 by $x-3$.
4. x^3-5x^2+9x-7 by $x-2$.
5. x^4-3x^2+2x-8 by $x-2$.
6. $3x^2-5x-9$ by $x-4$.
7. $x^2-2x-35$ by $x+5$.
8. x^3-3x^2+2x by $x+3$.
9. x^6+y^6 by x^2+y^2.
10. x^3+a^3 by $x-a$.

Prove that the following divisions are exact:

11. c^3+27 by $c+3$.

Put -3 in place of c in c^3+27, and $-27+27=0$.

12. $1-m^7$ by $1-m$.
13. $x^2-2x-15$ by $x-5$.
14. x^3+2x^2-x-2 by $x+1$.
15. x^5-a^5 by $x-a$.
16. x^4-a^4 by $x-a$.
17. x^8-a^8 by $x+a$.
18. x^9+a^9 by $x+a$.
19. x^6-a^6 by $x-a$.
20. $4x^3+13x^2-32x+15$ by $x+5$.

EXERCISES

138. 1. Explain the Law of Exponents in division.

2. What is the etymological meaning of "inverse"? Of "quotient"?

3. Express $(ab)^4$ as a product of powers.

4. What is the value of a^0b^0?

5. Give two methods by which division may be checked.

6. What advantage do you see in using detached coefficients in division? When should the method be used?

7. Why is it important to arrange both dividend and divisor according to the powers of some common letter?

8. Make an example illustrating the advantage of arranging the terms according to *ascending* powers of some common letter.

9. Divide $1-x+x^2$ by $1+x+x^2$ to 5 terms in the quotient. Write the next five terms.

10. What is the divisor if the quotient is $a^3 + 3a$, the dividend $a^5 - 8$, and the remainder $9a - 8$?

11. There are $16bc + 24c$ square feet in a hall $8c$ feet wide. What is the length?

12. Find value of $a - (b - c) + a\{[2c - (3b - 5a) \div c] + b\}$, when $a = 1$, $b = 3$, $c = 2$.

13. Divide $a^3 + b^3 + c^3 - 3abc$ by $a + b + c$, checking the quotient by symmetry.

14. Find the remainder when $x^4 - 3x^2 + x - 3$ is divided by $x - 1$.

15. Divide $a^4 + 2a^3b + 2a^2b + a^2b^2 + 4ab^2 + 2b^3$ by $a + b$; then divide the quotient by $a + b$. Would dividing by $(a + b)^2$, or $a^2 + 2ab + b^2$, give the same result?

16. Make an example to show that dividing by $x^2 - x - 6$ gives the same result as dividing successively by $x + 2$ and $x - 3$.

REVIEW EXERCISES

139. **1.** Add $2m + n - 3p$, $2n + p - 3m$, and $2p + m - 3n$.

2. Add $\frac{4}{3}a - \frac{3}{4}b + \frac{5}{2}c$, and $\frac{1}{2}a + \frac{2}{3}b - \frac{3}{5}c$.

3. From $2a^3 - 3a^2b + 2ab^2$ subtract $a^3 - ab^2 + b^3$.

4. From $\frac{1}{4}a^2 - \frac{1}{3}ab + \frac{1}{2}b^2$ subtract $\frac{1}{4}b^2 - \frac{1}{3}ab + \frac{1}{2}a^2$.

5. From $2y - 3[y - (x - y)]$ subtract $2x - 3[x - (y - x)]$.

6. From $3a^2 - 2a + 7$ subtract the sum of $a^2 - a + 9$ and $2a^2 - 7a - 6$.

7. From the product of $a + b$ and $a - b$ subtract $a^2 - b^2$.

8. What must be added to $x^2 + 3y^2 + 2z^2$ to produce $y^2 - 3x^2$?

9. Simplify $x - \{y + [x - (y + x)]\}$.

10. Simplify $7a - 2b - [(3a - c) - (2b - 3c)]$.

11. Simplify $x - (5x + 2y) - [3x - (-5x + 6y) - x]$.

12. Place within a parenthesis, preceded by the minus sign, the last three terms of $3ax + 4by - 5cx + 7dy - 8ex$.

Multiply:

13. $2a^3 - 3a^2 + 2a$ by $2a^2 + 3a + 2$.

14. $1 - x - 3x^2 - x^3$ by $1 + 2x + x^2$.

15. $a^4 + a^3 + a^2 + a + 1$ by $a - 1$.

16. $1 + y^6 - 2y^3$ by $y^2 + 1 - 2y$.

17. $a^{2m} - a^m b^m + b^{2m}$ by $a^m + b^m$.

18. $a^2 + b^2 + 1 - ab - a - b$ by $a + b + 1$.

Perform the indicated operations:

19. $(x - a)(x - b)(x - c)$.

20. $(a^4 + b^4)(a^2 + b^2)(a + b)(a - b)$.

21. $(x^m - y^m)(x^m + y^m)(x^n - y^n)$.

Multiply, using detached coefficients:

22. $x^4 + x^2 + 1$ by $1 - x^2 + x^4$.

23. $1 - 2x + 3x^2 + x^3$ by $2 - x + 2x^2$.

24. $3.2a^2 - 4.5ab + 1.8b^2$ by $1.5a - 3.5b$.

25. $3x^2y^2 + x^3y - x^4 + xy^3$ by $x^3 + x^2y - y^3$.

26. $x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$ by $1 + x^2 + x$.

Perform the divisions, checking results:

27. $a^4 + a^2b^2 + b^4$ by $b^2 - ab + a^2$.

28. 1 by $a + x$ to five terms in the quotient.

29. $x^4 - 2ax^3 + 2a^3x - a^4$ by $x^2 - a^2$.

30. $9x^5 - 6x^2 - 5x^4 + x^5 - x + 2$ by $x^2 - 3x + 2$.

31. $x^2 - 2xz - 4y^2 + 8yz - 3z^2$ by $x - 2y + z$.

32. $3.9x^2 - 4.1xy - 11.4y^2$ by $1.5x - 3.5y$.

33. $2x^7 + x^6 + 2x^5 + 1$ by $1 + x + x^2$.

34. $ax^2 - ab^2 + b^2x - x^3$ by $(x + b)(x - a)$.

35. $2x^{3n} - 6x^n y^n + 6x^n y^{2n} - 2y^{3n}$ by $x^n - y^n$.

Perform the divisions, using detached coefficients:

36. x^4+2x^2+x+2 by x^2+x+1.

37. x^5-5x^2+5x-1 by $1+x^2-2x$.

38. $x^6+6x^3-4x^2+24x-27$ by x^3+2x-3.

39. $.25x^4-1.8x^3+3.24x^2-12.25$ by $.5x^3-1.8x-3.5$.

40. $x^{10}+x^5+1$ by x^2+x+1.

41. a^8-10a^2-5a+4 by a^3-2a^2+3a-1.

42. $y^5-5y^4+9y^3-6y^2-y+2$ by y^2-3y+2.

43. $a^{2m}-b^{2n}+2b^nc^r-c^{2r}$ by $a^m-b^n+c^r$.

44. $a^4+b^4+c^4-2b^2c^2-2c^2a^2-2a^2b^2$ by $a+b+c$.

45. Divide $x^2-(a+b)x+ab$ by $x-b$.

46. Divide $1-x^6$ by $1+2x+2x^2+x^3$.

47. Divide the product of $2x^2-x-1$ and $3x^2+x-2$ by the product of $2x+1$ and $3x-2$.

48. Subtract the square of $a-(b-c)$ from its cube.

49. Show, without actually dividing, that x^5+y^5 is exactly divisible by $x+y$.

50. What is the dividend when the divisor is $x-a$ and the quotient $a^3-2ax+6$?

51. Simplify $a-\{2a-(b-6c)-[a-(-2b-5c)]\}$.

52. Simplify $(m^2-n^2)p-(m-n)[m(n+p)-n(m-p)]$.

53. Divide $n^{10}+n^9+n^8+2n^7+2n^4+2n^3+n^2+n+1$ by $n+1$.

54. Divide, without simplifying terms, $(x+n)^2-3(x+n)-28$ by $(x+n)-7$.

55. Divide $x^3+(a+b+c)x^2+(ab+ac+bc)x+abc$ by $x^2+(a+b)x+ab$.

56. The product of two polynomials is $24x^4-14x^3-9x^2+43x-84$. If one of them is $4x^2-3x+7$, what is the other?

57. Verify the identity $2(a^2+b^2+c^2-bc-ca-ab)\equiv(b-c)^2+(c-a)^2+(a-b)^2$.

58. Verify the identity $(a+b)^3\equiv a^3+b^3+3ab(a+b)$, checking for $a=1$, $b=2$.

EQUATIONS AND PROBLEMS

140. 1. Find the value of x in the equation $12x - 3 = 5x + 11$.

$$12x - 3 = 5x + 11.$$

Adding 3 and subtracting $5x$ (Axs. 2, 3), we have

$$12x - 5x = 11 + 3.$$

By uniting terms, $7x = 14.$

Dividing by 7, $x = 2.$

CHECK. Substituting 2 for x in the original equation,

$$24 - 3 = 10 + 11.$$

2. Find the value of x in the equation $5x - (2x + 5) = 7$.

$$5x - (2x + 5) = 7.$$

By removing the parentheses, $5x - 2x - 5 = 7.$

By adding 5, $5x - 2x = 7 + 5.$ (Ax. 2)

Uniting terms, $3x = 12.$

Dividing by 3, $x = 4.$ (Ax. 7)

CHECK. Give it.

3. Find the value of x in the equation $7x - 6(10 - x) = 33x$.

$$7x - 6(10 - x) = 33x.$$
$$7x - 60 + 6x = 33x.$$
$$7x + 6x - 33x = 60.$$
$$-20x = 60.$$
$$x = -3.$$

Explain how each equation is obtained from the preceding.

CHECK. Give it.

4. Find the value of x in the equation $x^2 - 4 = 5$.

$$x^2 - 4 = 5.$$
$$x^2 = 5 + 4 = 9.$$

Extracting the square root, $x = \pm 3.$

The sign $\pm$ before the 3, read *plus* or *minus*, shows that the root is *either* $+$ or $-$. For $(+3) \times (+3) = 9$, and $(-3) \times (-3) = 9$. The negative value may not always have a meaning in particular problems.

5. A man spends $\frac{1}{6}$ of his income for rent, $\frac{1}{5}$ for groceries, and has \$1140 left. What is his income?

Let x = the *number* of dollars in his income.

Then $\frac{x}{6}+\frac{x}{5}=\frac{11x}{30}$ = the number to be deducted.

$x-\frac{11x}{30}=1140$, by the conditions.

$\frac{19x}{30}=1140.$

$\frac{x}{30}=60$, by dividing by 19.

$x=1800.$

∴ the sum is \$1800.

CHECK. $1800-\frac{11}{30}$ of $1800=1140$.

6. The cost of an article is \$16, and this is 20% less than the marked price. What is the marked price?

Let x = the *number* of dollars of marked price.

Then $.20x$ = the number of dollars of discount.

$x-.20x=16.$ (Why?)

$.80x=16.$ (Why?)

$x=20.$ (Why?)

∴ the marked price is \$20.

CHECK. \$20 − $\frac{1}{5}$ of \$20 = \$16.

7. A father is now four times as old as his son; in 20 years he will be only twice as old. How old is each now?

Let x = the *number* of years in the son's age.

Then $4x$ = the number of years in the father's age.

$x+20$ = the number in the son's age 20 years hence.

$4x+20$ = the number in the father's age 20 years hence.

$2(x+20)=4x+20.$

$2x+40=4x+20$, by removing the parentheses.

$-2x=-20.$

$x=10$, by dividing by -2.

∴ the son is 10 years old, and the father is 40 years old.

8. A certain street contains 144 square rods, and the length is 16 times the width. Find the width.

Let x = the number of rods in the width of the street.
Then $16x$ = the number of rods in the length of the street.
$x \times 16x$ = the area of the street in square rods.
$16x^2 = 144.$
$x^2 = 9.$
$x = \pm 3.$

Therefore, the width of the street is 3 rods. The negative root has no meaning in this particular problem.

Find the value of x, and verify the results:

9. $5x - 3 = 22.$
10. $5x = 28 - 2x.$
11. $4x - 14 = x - 2.$
12. $-3x - 7 = -4x + 7.$
13. $5x^2 = 80.$
14. $17x - 10 = 15x + 40.$
15. $4x + (3x - 5) = 16.$
16. $1 - 4x^2 = x^2 - 4.$
17. $17 + 3(2x - 5) = 36.$
18. $x + 2 = \frac{2x}{3} + 3.$
19. $3x^2 + 1 = 2x^2 + 10.$
20. $2x - (5x + 5) = 7.$
21. $2(x + 3) - 3(x + 2) = -3.$
22. $4(x - 3) + 10 = 3(x + 4).$
23. $2x - 5 - (x - 3) = 2.$
24. $2x - 3(x - 3) + 2 = 0.$
25. $ax^2 + b = bx^2 + a.$
26. $x + 2(x - 5) = 5(13 - x) - 3.$
27. $3x - 100 = 200 - 8(60 - x).$
28. $x = \frac{x}{2} + \frac{x}{3} + 60.$
29. $5x - [4 - (6x - 3)] = 26.$

Solve the following problems, using proper checks:

30. If from 3 times a certain number we subtract 11, the result is 16. Find the number.

31. Three times a certain number, subtracted from 10, equals -5. Find the number.

32. If 5 is subtracted from twice a certain number, the result is 10 more than 9 times the number. Find the number.

33. What number is that from one third of which, if 7 is subtracted, the result is 11?

34. What sum of money loaned for 5 years at 6% amounts to $936?

35. A publisher received $160 as the proceeds of a sale of books. For how much were the books sold, the agent's commission being 20 per cent?

36. A room containing 200 square meters is twice as long as it is wide. Find the width.

37. What number is that which, if subtracted from 99, equals 23 more than 3 times the number?

38. The sum of three numbers is 50; the first is twice the second, and the third is 16 less than 3 times the second. Find the numbers.

39. A man is 30 years older than his son, and 10 years ago he was 3 times as old. What is the age of each?

40. If a certain number is diminished by 6 and the remainder multiplied by 6, the result is the same as if the number were diminished by 4 and the remainder multiplied by 4. Find the number.

41. Double a certain number exceeds its third by 20. Find the number.

42. A rectangular lot contains an acre, and its width is $\frac{4}{5}$ of its length. What is its width?

43. There are two numbers, of which the sum is 110, and the first number increased by 4 equals 3 times the second diminished by 2. What are the numbers?

44. If it takes a boy 7 times as long to walk up a hill as to coast down, and the time of a round trip is 16 minutes, what is the time required to coast down?

45. A triangular field contains 5 hectares, and its altitude is $\frac{4}{5}$ of its base. What is the base?

46. A pupil was told to double a certain number, then to add the original number, and finally to subtract 20. He gave as the result 25. What was the original number?

47. Distribute \$100 among Miller, Dick, and Berlin, so that Miller shall have \$20 more than Dick, and Dick shall have \$8 less than Berlin.

48. If I read a volume of 350 pages in 5 days, reading 10 pages more each day than the preceding day, how many pages do I read the first day?

49. The area of one square field is twice that of another, and they together contain 867 square rods. Find the length of a side of the smaller.

50. In 16 years Ralph will be 3 times as old as he was 2 years ago. What is his age?

51. Ray is 3 times as old as James, but 10 years hence he will only be twice as old. Find the age of each.

52. Find three consecutive numbers such that if they be diminished by 10, 17, and 26 respectively, their sum will be 10.

53. A debt of \$102 is paid in ten-dollar, five-dollar, and two-dollar bills, of each an equal number. How many bills of each kind were used?

54. A has \$250, and B has \$75. How much must A give B, that each may have the same sum?

55. Six pounds of coffee and 5 pounds of tea cost \$6.40. The tea costs twice as much as the coffee. What is the price of each a pound?

56. The time past 12 o'clock equals $\frac{1}{3}$ of the time before 1 o'clock. What time is it?

57. A ball, bat, and mask cost \$3. If the mask cost as much as the ball and bat together, and the ball 5 times as much as the bat, what did each cost?

---

58. A number of men were to raise a certain fund by paying \$2 each, but 5 men failed to contribute, and in consequence each of the actual contributors had to pay \$3 each. How many contributors were there?

59. I can buy 8 meters of cloth of one kind, or 6 meters of another kind for the same money. If the latter costs $1 a meter more than the former, how much does a meter of the former cost?

60. A purse contains 3 times as many dollars as quarters; if 8 dollars and 8 quarters be taken away, there will be 5 times as many dollars as quarters. Find the number of pieces of each kind.

61. The ages of three brothers differ successively by 4 years. If the sum of their ages is 54 years, what is the age of each?

62. Fourteen persons were to raise a certain fund, but 4 of them failed to contribute, and in consequence each of the others had to contribute $4. What was the amount each of the original 14 persons was to pay?

63. If 2 is subtracted from a certain number, and the difference is subtracted from 30, the result is 3 times the number. Find the number.

64. A newsboy has $6 in silver half dollars and quarters, and has 20 coins in all. How many of each has he?

65. Divide 18 into two such parts that 5 times the greater increased by 4 shall be equal to 9 times the less diminished by 4.

66. A dealer bought a lot of pigs at $1 each. After 9 of them died, he sold the remainder at $2 each and made a profit of $7. How many did he buy?

67. A had $16; after spending a certain sum he found that he had left 3 times as much as he had spent. How much did he spend?

68. Twice a certain number added to 24 exceeds 50 by as much as the number itself is less than 100. Find the number.

69. How long will it take a sum of money to double itself at 6% simple interest?

70. A dealer marked a bicycle to sell at 10% profit, but was obliged to sell it for $15 less than the marked price. What was the cost of the bicycle if he lost 5% on the cost?

71. A invested a certain sum at 5% interest and B an equal sum at 3%. The annual income from A's money was $400 more than the income from B's money. What was the capital invested by each?

72. I owe $1302, which is to be paid in a year. What cash payment to-day will pay the debt if money is worth 5%?

73. I bought 3 pounds of rice and 10 pounds of sugar for $1.24. The rice cost 2 cents a pound more than the sugar. What was the cost of each a pound?

74. A and B started in business with equal capital. A doubled his money and then made $100, while B tripled his money and then lost $400, when it was found they had equal sums. What was the original capital of each?

75. A dealer bought 200 bushels of apples for $260. After retaining a part for his own use, he sold the rest at a profit of 40%, clearing $13 on the entire cost. How many bushels did he keep?

76. The length of a rectangle is 6 feet longer and its width 5 feet shorter than the side of an equivalent square. Find its length.

77. Roy and Guy each have the same number of marbles. After Roy loses 100 and Guy loses 20, the number Roy has left is to the number Guy has left as 2 to 3. How many had each at first?

78. Atmospheric air is a mixture of 4 parts by volume of nitrogen with 1 of oxygen. How many cubic feet of oxygen are there in a room 30 feet long, 15 feet wide, and 12 feet high?

79. A tank holds three times as much water as a smaller cistern. They are filled by a pipe supplying 6000 liters each day. If no water is drawn off, the tank and cistern will be filled in 16 hours. What is the capacity of each?

FACTORS

141. The *rational integral expressions* which multiplied together will produce a given rational integral expression are called the **Factors** of that expression.

Thus, in the expression $2x(x+1)(x^2+x+1)$, 2 is a numerical factor, x is a monomial factor, $x+1$ is a binomial factor, and x^2+x+1 is a trinomial factor.

142. To *factor* a rational integral expression is to discover all its component rational integral factors.

An expression that has no factors is said to be **Prime.**

In factoring, the word "expression" is limited to mean only rational integral expressions. Although $x-1 \equiv (\sqrt{x}+1)(\sqrt{x}-1)$ and

$$x - \frac{1}{x^2} \equiv \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x}\right),$$

these expressions are not factorable in the *present sense*, because the factors of the first are not rational, and neither the expression nor the factors are integral in the second.

143. Factoring is seen to be the *inverse* of multiplication, and it depends chiefly on the recognition of expressions as *products.* As an aid to this end certain identities already considered are used as type forms with which the given expressions are compared in discovering the factors.

Thus, from an inspection of the expression $a^2 - b^2$, we recognize it as the common product

$$(a+b)(a-b) \equiv a^2 - b^2,$$

and we know at once that the factors are $a+b$ and $a-b$.

NOTE. Although the use of identities is involved in factoring, it may be found advisable in practice for the student to employ the symbol $=$ instead of $\equiv$ because more easily written.

144. Monomial Factors.

From multiplication we have

$$(a + b - c)\,m \equiv ma + mb - mc.$$

This identity may be written

$$ma + mb - mc \equiv m(a + b - c). \qquad \text{(Why ?)}$$

In the latter form it is readily seen that polynomials of the form $ma + mb - mc$ can be factored by writing the *common* monomial factors before a parenthesis containing the other factors of each term.

Thus, $ab + ac \equiv a(b + c)$.

145. When all the terms of a given expression do not contain a common monomial factor, it is often possible to group the terms so that all the groups shall have a common factor, and then to factor the resulting expression.

Thus,
$$\begin{aligned} x^2 - xy + xz - yz &\equiv (x^2 - xy) + (xz - yz) \\ &\equiv x(x - y) + z(x - y) \\ &\equiv (x + z)(x - y). \end{aligned}$$

In the first step each group is factored, when a common factor $(x - y)$ appears. Then x times this factor $+ z$ times the same $= (x + z)$ times the factor. Note that $(x - y)$ is treated as a monomial.

1. Factor the expression $4x^3 + 8x^2 - 16x$.

$$4x^3 + 8x^2 - 16x \equiv 4x(x^2 + 2x - 4).$$

CHECK. Let $x = 1$; then $-4 = 4(-1)$.

2. Factor the expression $mx - my - nx + ny$.

$$\begin{aligned} mx - my - nx + ny &\equiv (mx - my) - (nx - ny) \\ &\equiv m(x - y) - n(x - y) \\ &\equiv (m - n)(x - y). \end{aligned}$$

CHECK. Let $x = 2$, $y = 1$, $m = 4$, $n = 3$; then $8 - 4 - 6 + 3 = 1 \times 1$.

Factor the following expressions:

3. $2a^2 + 6ab$.
4. $3x^2 - 15x$.
5. $9y^3 + 6ay^2$.
6. $x^{2n} + 2x^n$.

7. $12\,a^2b^3 - 16\,ab^4 + 8\,a^2b^5$.
8. $a^2b^2c^2 - ab^2c^3 + a^2b^3c$.
9. $x^2 - xz + xy - xyz$.
10. $a^3b + ab^2c + a^2c$.
11. $x^3 - 5\,x^2y + 20\,x^2y^2$.
12. $3\,cx^2 + 6\,c^2x - 9\,c^3x^3$.
13. $2\,a^2b^3 - 6\,x^2b^2 + 2\,by$.
14. $2(s+1)^2 - 4(s+1)$.
15. $mx - 2\,my - nx + 2\,ny$.
16. $6\,a^3 + 4\,a^2 - 9\,a - 6$.
17. $27\,m^4n + 18\,m^3n^2 + 3\,m^2n^3$.
18. $m^3x + mn^2x - m^2y - n^2y$.
19. $2\,x^ny^4 - 8\,x^{2n}y^3 + 6\,x^{3n}y^2$.
20. $m^2(a-x) - b^2(x-a)$.
21. $6\,x^2 - 9\,ax + 4\,bx - 6\,ab$.
22. $3\,ax - ay - 9\,bx + 3\,by$.
23. $3\,x^3y^2 - 6\,x^4y^2 + 9\,x^2y^4 - 12\,x^3y^3$.
24. $15\,m^3n^3p + 30\,m^3n^2p - 25\,m^2np^2$.
25. $m^4 - m^3n^2 + m^2n - mn^3 + n^5 - mn^3$.
26. $2\,mx - \frac{1}{2}\,x^2$.
27. $am^2 - \frac{1}{2}\,a^2m$.
28. $\frac{1}{4}\,x^2 - \frac{1}{6}\,xy$.
29. $n^{12} + n^{11} - n^{10} - n^9$.
30. $am + an - bm - bn - b + a$.
31. $r^2x - r^2y - ry - y + x + rx$.

146. Trinomials that are Squares of Binomials.

From Art. 118, I and II, we have

$$a^2 + 2\,ab + b^2 \equiv (a+b)^2;$$
$$a^2 - 2\,ab + b^2 \equiv (a-b)^2.$$

From these identities it is readily seen that any trinomial that is the square of a binomial can be factored directly from this type. That is, a trinomial, two of whose terms are the squares of two numbers respectively, and the remaining term is twice their product, can be written as the square of the sum of the numbers, or as the square of the difference of the numbers, according as the sign of the remaining term of the trinomial is plus or minus.

147. Since $(a+b)^2 \equiv a^2 + 2\,ab + b^2$, and $(a-b)^2 \equiv a^2 - 2\,ab + b^2$, it follows that we may express these results in the form $(a \pm b)^2 \equiv a^2 \pm 2\,ab + b^2$, where the **Double Sign**, $\pm$, indicates that either the sign $+$ or the sign $-$ may be taken, provided either the upper or the lower sign is kept throughout.

1. Factor $x^2 + 10\,x + 25$.

$$x^2 + 10\,x + 25 \equiv (x + 5)^2.$$

CHECK. Let $x = 1$; then $36 = 6^2$.

2. Factor $9\,a^2 - 24\,ab + 16\,b^2$.

$$9\,a^2 - 24\,ab + 16\,b^2 \equiv (3\,a - 4\,b)^2.$$

CHECK. Let $a = 1$, $b = 1$; then $1 = (-1)^2$, or 1.

Factor the following expressions:

3. $a^2 + 12\,a + 36$.
4. $a^2 - 16\,a + 64$.
5. $a^4 - 2\,a^2b + b^2$.
6. $4\,a^2 + 4\,ab + b^2$.
7. $2\,a^2x^2 + a^4 + x^4$.
8. $m^2x^2 - 2\,mxy + y^2$.
9. $36\,a^4 - 12\,a^2b^2 + b^4$.
10. $x^2y^4 - 6\,xy^2 + 9$.
11. $x^2 - 10\,x + 25$.
12. $a^2 - 4\,ab + 4\,b^2$.
13. $a^2 + 6\,ab + 9\,b^2$.
14. $4\,a^2 - 12\,ab + 9\,b^2$.
15. $25\,a^2b^2 - 30\,ab + 9$.
16. $a^6 - 20\,a^3 + 100$.
17. $x^2y^4 + 18\,xy^2z + 81\,z^2$.
18. $1 - 8\,a^2 + 16\,a^4$.
19. $4\,x^4 + 20\,x^2y^2 + 25\,y^4$.
20. $a^{2n} + 2\,a^nb + b^2$.
21. $1 - 6\,a^3b + 9\,a^6b^2$.
22. $a^{2n} - 2\,a^nb^nc^n + b^{2n}c^{2n}$.
23. $a^{2n}x^{4n} - 2\,a^nb^nx^{2n}y^{3n} + b^{2n}y^{6n}$.
24. $(a + c)^2 + 2(a + c) + 1$.

The method of factoring is the same whether both squares are monomials or not. Whenever a polynomial appears as one of the squares it is treated as a monomial.

25. $(a + b)^2 - 4(a + b) + 4$.
26. $(x - y)^2 + 2(x - y) + 1$.
27. $(m^2 + n^2)^2 - 2(m^2 + n^2) + 1$.
28. $x^2 + 2\,x(x - y) + (x - y)^2$.

Sometimes the trinomial that is a square does not appear until the monomial factors are removed.

Thus, $2\,ax^2 - 4\,ax + 2\,a \equiv 2\,a(x^2 - 2\,x + 1) \equiv 2\,a(x - 1)^2$.

29. $a - 20\,ab + 100\,ab^2$.
30. $3\,x^2 - 30\,x + 75$.
31. $m + 4\,m^2 + 4\,m^3$.
32. $2\,x^3 + 12\,x^2y + 18\,xy^2$.
33. $27\,c^2 - 90\,c + 75$.
34. $5\,a^2y - 20\,aby + 20\,b^2y$.
35. $(x^2 - 1)^2 + 6\,x(x - 1) + 9$.
36. $(b - 1)^2 + 4\,a(b - 1) + 4\,a^2$.

148. The Difference of Two Squares.

From Art. 118, III, we have

$$a^2 - b^2 \equiv (a + b)(a - b).$$

It follows from this identity that expressions which are in the form of this type, or which can be reduced to such a form, can be factored. That is, the difference of the squares of two numbers can be written as the product of the sum and the difference of the numbers.

The difference of any even powers of two numbers can be written as the difference of two squares and factored in a similar manner.

Thus, $a^6 - b^6$ may be written $(a^3)^2 - (b^3)^2$.

1. Factor $x^2 - 25$.

$$x^2 - 25 \equiv (x + 5)(x - 5).$$

CHECK. Let $x = 1$; then $1 - 25 = 6 \times (-4)$.

2. Factor $x^4 - y^4$.

$$\begin{aligned} x^4 - y^4 &\equiv (x^2)^2 - (y^2)^2 \\ &\equiv (x^2 + y^2)(x^2 - y^2) \\ &\equiv (x^2 + y^2)(x + y)(x - y). \end{aligned}$$

CHECK. Let $x = 2$, $y = 1$; then $16 - 1 = 5 \times 3 \times 1$.

It will be observed that sometimes one or more of the factors of a number may be factored.

Factor the following expressions:

3. $x^2 - 9$.	11. $9z^2 - 36x^2y^2$.	19. $m^8 - n^8$.
4. $m^2 - n^2$.	12. $m^2n^2 - 1$.	20. $a^2x^{4n} - z^4$.
5. $x^2 - 9b^2$.	13. $z^4 - 9x^4y^4$.	21. $x^{2n} - y^{2n}$.
6. $9a^2 - 16$.	14. $16x^2 - 9y^2$.	22. $a^6 - b^4$.
7. $4y^2 - x^2$.	15. $64a^2 - 81b^2$.	23. $x^2y^2 - 4x^{2n}$.
8. $1 - 9y^2$.	16. $m^2n^2 - 4a^2$.	24. $x^{2n} - 1$.
9. $x^4 - 25$.	17. $1 - a^{2n}$.	25. $a^4 - 16$.
10. $x^4 - 49y^4$.	18. $x^8 - 1$.	26. $1 - 121x^2$.

27. $a^2 + 2\,ab + b^2 - c^2$.

$$a^2 + 2\,ab + b^2 - c^2 \equiv (a + b)^2 - c^2$$
$$\equiv (a + b + c)(a + b - c).$$

CHECK. Let $a = b = c = 1$; then $3 = 3 \times 1$.

It is seen that a polynomial may be factored under this type when the terms can be combined so as to form two squares connected by the *minus* sign, one or both of them being polynomials.

Thus, $x^2 - a^2 + y^2 - b^2 + 2\,ab - 2\,xy \equiv (x - y)^2 - (a - b)^2$.

28. $x^2 + y^2 - a^2 + 2\,xy$.

29. $x^2 - 1 + y^2 + 2\,xy$.

30. $n^2 - a^2 - 4\,n + 4$.

31. $a^2 + b^2 - 4 - 2\,ab$.

32. $m^2 - 16\,n^2 - 2\,mx + x^2$.

33. $(a - 2\,b)^2 - 4\,c^2$.

34. $d^2 - 1 + c^2 + 2\,cd$.

35. $4\,m - 1 - 4\,m^2 + 4\,a^2$.

36. $1 - n^2 - 4\,q + 4\,q^2$.

37. $9\,x^2 - a^2 - b^2 + 2\,ab$.

38. $(a + b)^2 - (c + d)^2$.

39. $(1 - n)^2 - (m - n)^2$.

40. $(m + n - 1)^2 - 9$.

41. $(x - 2\,y)^2 - (x - 5)^2$.

42. $(a + b + c)^2 - (a - b - c)^2$.

43. $x^2 - 2\,x - a^2 - b^2 + 2\,ab + 1$.

44. $2\,xy - a^2 + x^2 - 2\,ab - b^2 + y^2$.

45. $(x^2 + xy - y^2)^2 - (x^2 - xy - y^2)^2$.

46. $m^2 + n^2 - p^2 - q^2 + 2(mn + pq)$.

47. $a^4 + a^2b^2 + b^4$; also $a^4 - 7\,a^2 + 1$.

$$a^4 + a^2b^2 + b^4 \equiv a^4 + 2\,a^2b^2 + b^4 - a^2b^2$$
$$\equiv (a^2 + b^2)^2 - a^2b^2$$
$$\equiv (a^2 + b^2 + ab)(a^2 + b^2 - ab).$$

Notice that the given expression is brought to the given type form by adding a^2b^2 to the second term to make the expression the square of $a^2 + b^2$, and then subtracting a^2b^2 to restore the identity. Any trinomial of the form $a^4 \pm a^2b^2 + b^4$ is a special case of this type.

$$a^4 - 7\,a^2 + 1 \equiv a^4 - 7\,a^2 + 1 + 9\,a^2 - 9\,a^2$$
$$\equiv a^4 + 2\,a^2 + 1 - 9\,a^2$$
$$\equiv (a^2 + 1)^2 - 9\,a^2$$
$$\equiv (a^2 + 1 + 3\,a)(a^2 + 1 - 3\,a).$$

If we *add* $9\,a^2$, the expression becomes $a^4 + 2\,a^2 + 1$, which is a square. We must then *subtract* $9\,a^2$ to restore the identity.

48. $a^4 - 7\,a^2b^2 + b^4$.
49. $m^4 + m^2n^2 + n^4$.
50. $x^4 - 3\,x^2 + 1$.
51. $x^4 + 7\,x^2 + 16$.
52. $x^4 + 2\,x^2 + 9$.
53. $a^4 + 2\,a^2b^2 + 9\,b^4$.
54. $m^4 - 8\,m^2n^2 + 4\,n^4$.
55. $4\,a^4 - 21\,a^2b^2 + 9\,b^4$.

149. Cubes of Binomials.

From Art. 118, IV, V, we have

$$a^3 + 3\,a^2b + 3\,ab^2 + b^3 \equiv (a + b)^3;$$
$$a^3 - 3\,a^2b + 3\,ab^2 - b^3 \equiv (a - b)^3.$$

It therefore appears that the factors of expressions of this form can be written out at once if we but recognize the polynomial as a cube.

1. Factor $8\,x^3 + 36\,x^2y + 54\,xy^2 + 27\,y^3$.

$$8\,x^3 \equiv (2\,x)^3, \text{ and } 27\,y^3 \equiv (3\,y)^3.$$
$$36\,x^2y \equiv 3 \times (2\,x)^2 \times 3\,y.$$
$$54\,xy^2 \equiv 3 \times 2\,x \times (3\,y)^2.$$

Hence the given polynomial is a cube, and

$$8\,x^3 + 36\,x^2y + 54\,xy^2 + 27\,y^3 \equiv (2\,x + 3\,y)^3.$$

CHECK. Let $x = y = 1$; then $125 = 5^3$.

2. Factor $27\,a^9 - 27\,a^6 + 9\,a^3 - 1$.

$$27\,a^9 - 27\,a^6 + 9\,a^3 - 1 \equiv (3\,a^3)^3 - 3(3\,a^3)^2 + 3(3\,a^3) - 1$$
$$\equiv (3\,a^3 - 1)^3.$$

CHECK. Let $a = 1$; then $8 = 2^3$.

Factor the following expressions:

3. $x^3 + 3\,x^2 + 3\,x + 1$.
4. $8\,a^3 - 12\,a^2 + 6\,a - 1$.
5. $m^3 + 6\,m^2 + 12\,m + 8$.
6. $8\,b^3 - 12\,b^2 + 6\,b - 1$.
7. $a^3 - 6\,a^2b + 12\,ab^2 - 8\,b^3$.
8. $x^3 - 3\,x^2 + 3\,x - 1$.
9. $8\,x^6 - 12\,x^4 + 6\,x^2 - 1$.
10. $8\,x^6 - 12\,x^4y + 6\,x^2y^2 - y^3$.
11. $a^3 - 12\,a^2b + 48\,ab^2 - 64\,b^3$.
12. $8\,x^3 - 36\,x^2a + 54\,xa^2 - 27\,a^3$.
13. $a^3x^3 + 3\,a^2bx^2y + 48\,a^2b^2xy^2 + b^3y^3$.
14. $x^3y^3 - 12\,abx^2y^2 + 48\,a^2b^2xy - 64\,a^3b^3$.

150. Trinomials of the Type $x^2 + px + q$.

When a trinomial is the product of two binomials, it can be resolved into two factors, whether equal or unequal. We have already had the form $x^2 + 2xy + y^2 \equiv (x + y)^2$. It is now necessary to consider the more general type $x^2 + px + q$.

$$(x + 2)(x - 3) \equiv x^2 + 2x - 3x - 6 \equiv x^2 - x - 6.$$
$$(x - 2)(x + 3) \equiv x^2 - 2x + 3x - 6 \equiv x^2 + x - 6.$$
$$(x + 2)(x + 3) \equiv x^2 + 2x + 3x + 6 \equiv x^2 + 5x + 6.$$
$$(x - 2)(x - 3) \equiv x^2 - 2x - 3x + 6 \equiv x^2 - 5x + 6.$$

In these products note that:

(*a*) The first term is the square of the term common to the factors.

(*b*) The second term is this common term, and its coefficient is the *algebraic* sum of the last terms of the factors.

(*c*) The last term is the product of the last terms of the factors, and it is positive when the last terms have like signs, otherwise it is negative.

Hence in factoring them —

(*a*) How may the first term of each factor be found?

(*b*) The last terms of the factors must be numbers whose (algebraic) *sum* is the coefficient of x, and whose *product* is the third term of the expression.

If the expression $x^2 + px + q$ has binomial factors, the first term of each must evidently be x. Assuming that a and b are the other terms of the factors, we have

$$x^2 + px + q \equiv (x + a)(x + b)$$
$$\equiv x^2 + (a + b)x + ab, \text{ by multiplication.}$$

It therefore appears that any trinomial of the form $x^2 + px + q$ can be written in the form $x^2 + (a + b)x + ab$ and factored by means of this identity, provided two numbers, a and b, can be determined such that their *sum* shall be p and their *product* q.

1. Factor $x^2 + 9x + 20$.

Here $a + b = p = 9$, and $ab = q = 20$. The only two numbers whose sum is 9 and whose product is 20 are 4 and 5. Hence

$$x^2 + 9x + 20 \equiv x^2 + (4 + 5)x + 4 \times 5$$
$$\equiv (x + 4)(x + 5).$$

CHECK. Let $x = 1$; then $30 = 5 \times 6$.

2. Factor $x^2 - 8x + 15$.

Here $a + b = -8$, and $ab = 15$. Since the product is ***positive***, both factors must be positive or both negative; and since their sum is ***negative***, we know they must both be negative. Is -8 the sum of -3 and -5? Is 15 their product? Then

$$x^2 - 8x + 15 \equiv x^2 + [(-3) + (-5)]x + (-3 \times -5)$$
$$\equiv (x - 3)(x - 5).$$

CHECK. Let $x = 1$; then $8 = -2 \times -4$.

3. Factor $m^2x^2 - 7mx - 18$.

Here $-7 = -9 + 2$, and $-18 = -9 \times 2$.

$\therefore (mx)^2 - 7(mx) - 18 \equiv (mx - 9)(mx + 2)$.

CHECK. Let $mx = 1$; then $-24 = -8 \times 3$.

How do we know that one of the second terms of the factors is positive and the other negative? Is their algebraic sum positive or negative? Then which has the greater absolute value? If both were positive, what would the second term of the trinomial be?

4. Resolve $x^2 + 7xy + 10y^2$ into two binomial factors.

Here $a + b = p = 7y = 2y + 5y$,

and $ab = q = 10y^2 = 2y \times 5y$.

$\therefore x^2 + 7xy + 10y^2 \equiv (x + 2y)(x + 5y)$.

5. Factor $a^2 + 12abx - 28b^2x^2$.

Here $a + b = p = 12bx = 14bx - 2bx$,

and $ab = q = -28b^2x^2 = 14bx \times (-2bx)$.

$\therefore a^2 + 12abx - 28b^2x^2 \equiv (a + 14bx)(a - 2bx)$.

CHECK. Let $a = bx = 1$; then $-15 = 15 \times -1$.

How do we know that the greater number ($14bx$) is positive? How do we know that the two factors have a common first term?

Factor the following expressions:

6. $x^2 + 5x + 6$.
7. $x^2 + 6x - 27$.
8. $c^2 - 7c + 10$.
9. $a^2 + 15a + 56$.
10. $a^2 - 3a - 18$.
11. $a^2 - 7a + 12$.
12. $x^2 - 4x - 32$.
13. $m^4 - 15m^2 - 100$.
14. $a^2 - 20ab - 96b^2$.
15. $p^2 + 12p - 85$.
16. $r^2 - r - 56$.
17. $x^4 + 8x^2 + 7$.
18. $a^2 + 3ab - 4b^2$.
19. $n^2x^2 - 21nx + 80$.
20. $x^2 - 13x + 42$.
21. $y^4 - 24y^2 + 63$.
22. $y^2 - 2y - 99$.
23. $x^6 + 7x^3 - 8$.
24. $a^2 - 8an - 20n^2$.
25. $c^4 - 4a^2c^2 + 3a^4$.
26. $a^6b^2 - a^3b - 2$.
27. $x^{2n} - x^n - 56$.
28. $a^{4n} - 3a^{2n} - 70$.
29. $5x^2 - 55x + 150$.
30. $a^2 + (m+n)a + mn$.
31. $x^2 + (c-d)x - cd$.
32. $(x+y)^2 + 9(x+y) + 8$.
33. $(m-n)^2 - 3(m-n) - 4$.

151. Trinomials of the Form $ax^2 + px + q$.

This type is more general than the one last considered. In the present case the binomial factors do not have a common first term. If we multiply $4x + 5$ by $2x + 3$, we get $8x^2 + 22x + 15$. Hence

$$8x^2 + 22x + 15 \equiv (4x + 5)(2x + 3).$$

In this particular case it will be observed that

$8x^2 \equiv 4x \times 2x$, the first terms of the factors;

$15 \equiv 5 \times 3$, the second terms; and

$22x \equiv 5 \times 2x + 3 \times 4x$, the sum of the *cross products*, as shown in the following arrangement:

$$\begin{array}{r} 4x + 5 \\ \times \\ 2x + 3 \\ \hline 10x + 12x = (10 + 12)x = 22x. \end{array}$$

These conditions must exist in the general type ax^2+px+q, if the trinomial is factorable. Assuming mx and nx to be the factors of the first term of the given type, and a and b the factors of the last term, we have

$$\begin{aligned} ax^2 + px + q &\equiv (mx + a)(nx + b) \\ &\equiv mnx^2 + (mb + na)x + ab. \end{aligned}$$

Note that $mb + na$ is the coefficient of x; also that $mb \times na \equiv mn \times ab$, the product of the last term and the coefficient of x^2.

It therefore appears that a trinomial of the form $ax^2 + px + q$ can be written in the form $mnx^2+(mb+na)x+ab$ and factored, provided two numbers can be determined such that their *sum* shall be the *coefficient* of x, and their *product* the *product* of the *last term* and the *coefficient* of x^2.

Thus to write $8x^2 + 22x + 15$ in the required form we must separate 22 into two parts such that their product shall be 8×15, or 120. These parts are 10 and 12. Hence we have $8x^2 + (10 + 12)x + 15$, or $8x^2 + 10x + 12x + 15$.

1. Factor $6x^2 + 19x + 10$.

We must separate 19 into two parts whose product is 6×10, or 60. These parts are 15 and 4.

$$\begin{aligned} \therefore\ 6x^2 + 19x + 10 &\equiv 6x^2 + 15x + 4x + 10 \\ &\equiv 3x(2x + 5) + 2(2x + 5) \\ &\equiv (3x + 2)(2x + 5). \end{aligned}$$

CHECK. Let $x = 1$; then $35 = 5 \times 7$.

2. Factor $6x^2 + x - 12$.

Here $6 \times (-12) = -72$, and the coefficient of x is 1. Hence the (algebraic) sum of the required numbers is $+1$, and their product is -72. They are therefore 9 and -8.

$$\begin{aligned} \therefore\ 6x^2 + x - 12 &\equiv 6x^2 + 9x - 8x - 12 \\ &\equiv 3x(2x + 3) - 4(2x + 3) \\ &\equiv (3x - 4)(2x + 3). \end{aligned}$$

CHECK. Let $x = 1$; then $-5 = -1 \times 5$.

Factor the following expressions:

3. $6x^2 - 19x + 15$.
4. $3a^2 + 11a + 6$.
5. $5a^2 + 32a - 21$.
6. $10y^2 + 13y - 3$.
7. $2m^2 + 5m + 2$.
8. $6a^2 - a - 12$.
9. $2y^2 + y - 15$.
10. $3a^2 - 17a + 10$.
11. $4x^2 - 4x - 35$.
12. $3x^2 + 13x + 12$.
13. $3n^2 - 2n - 5$.
14. $2c^2 + 7c + 6$.
15. $3x^2 + 5x - 12$.
16. $20x^2 + 37x + 8$.
17. $m^4 - 13m^2 + 36$.
18. $2x^4 + x^2 - 28$.
19. $9a^2 + 43a - 10$.
20. $4 - 9x - 9x^2$.
21. $1 - 5x + 4x^2$.
22. $3x^2 - 5xy - 2y^2$.
23. $40 + 2m - 2m^2$.
24. $5a^8 + 11a^4 - 12$.
25. $9x^{4n} - 4x^{2n} - 5$.
26. $(m+n)^2 + 5(m+n) - 24$.

152. Binomials.

We have already seen that binomials of the form $a^2 - b^2$ can be factored. By actual multiplication we have

$$(a+b)(a^2 - ab + b^2) \equiv a^3 + b^3,$$
$$(a-b)(a^2 + ab + b^2) \equiv a^3 - b^3.$$

It therefore appears that binomials of the form $a^3 + b^3$ and $a^3 - b^3$ can also be factored. It remains to consider the general types $a^n + b^n$ and $a^n - b^n$.

153. It was shown in Art. 137 that

$a^n + b^n$ is divisible by $a + b$ when n is an odd integer,

$a^n + b^n$ is divisible by $a - b$ never,

$a^n - b^n$ is divisible by $a + b$ when n is an even integer,

$a^n - b^n$ is divisible by $a - b$ always.

Hence we can tell at once whether expressions of this type have a binomial factor of the form $a + b$ or $a - b$.

154. If any expression of this form is factorable, we can readily write out the factors from the following quotients already found in Art. 132, III:

$$(a^n + b^n) \div (a + b) \equiv a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots + b^{n-1},$$
$$(a^n - b^n) \div (a + b) \equiv a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \cdots - b^{n-1},$$
$$(a^n - b^n) \div (a - b) \equiv a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}.$$

Thus, from the first identity we readily write out the quotient of $(a^5 + b^5) \div (a + b)$ as $a^4 - a^3b + a^2b^2 - ab^3 + b^4$. We therefore know that the factors of $a^5 + b^5$ are $a + b$ and $a^4 - a^3b + a^2b^2 - ab^3 + b^4$.

155. When n is even, expressions of the type $x^n - y^n$ should first be factored as the difference of two squares, as in Art. 148.

Thus, $a^{10} - b^{10} \equiv (a^5 - b^5)(a^5 + b^5) \equiv$, etc.

1. Factor the expression $27\,x^6 - y^6$.

$$27\,x^6 - y^6 \equiv (3\,x^2)^3 - (y^2)^3$$
$$\equiv (3\,x^2 - y^2)[(3\,x^2)^2 + 3\,x^2y^2 + (y^2)^2]$$
$$\equiv (3\,x^2 - y^2)(9\,x^4 + 3\,x^2y^2 + y^4).$$

CHECK. Let $x = y = 1$; then $27 - 1 = 2(9 + 3 + 1)$.

It will be observed that $(3\,x^2)^3 - (y^2)^3$ is identical in form with $a^3 - b^3$, a being represented by $3\,x^2$, and b by y^2.

2. Factor $8(x - y)^3 - 1$.

$$8(x - y)^3 - 1 \equiv [2(x - y) - 1][4(x - y)^2 + 2(x - y) + 1]$$
$$\equiv (2\,x - 2\,y - 1)(4\,x^2 - 8\,xy + 4\,y^2 + 2\,x - 2\,y + 1).$$

CHECK. Let $x = y = 1$; then $0 - 1 = -1 \times 1$.

3. Write the factors of $x^7 - y^7$.

From Art. 154,

$$x^7 - y^7 \equiv (x - y)(x^6 + x^5y + x^4y^2 + x^3y^3 + x^2y^4 + xy^5 + y^6).$$

Factor the following expressions:

4. $x^3 - 1$.
5. $a^5 - b^5$.
6. $m^3 - n^3$.
7. $27 - x^3$.
8. $1 - r^3$.
9. $m^4 - n^4$.
10. $x^5 + y^5$.
11. $r^6 - s^6$.
12. $m^7 + n^7$.
13. $p^8 - q^8$.
14. $a^3 - b^6$.
15. $y^3 + 27$.
16. $x^8 - 1$.
17. $8\,a^3 - b^3$.
18. $64 + 8\,a^3$.

19. $a^{10} - 1$.
20. $x^9 - y^9$.
21. $8a^3 + 8b^3$.
22. $x^7 - 128$.
23. $32a^5 + b^5$.
24. $a^{12} - c^9$.
25. $a^3b^3c^3 - 1$.
26. $a^3b^3c^6 + 8$.
27. $m^3n^3 - 27x^3$.
28. $125 - 8p^6q^6$.
29. $2^{2n} - y^{2n}$.
30. $y^{3n} - z^{3n}$.
31. $(a+b)^3 - c^3$.
32. $1 - (m-n)^3$.
33. $(a-b)^3 + c^3$.
34. $(x-y)^3 - (x+y)^3$.
35. $(m+n)^3 - (m-n)^3$.
36. $(x+a)^3 + 1000$.
37. $m^3 - (a-2)^3$.
38. $(a+b)^6 - (a-b)^6$.
39. $(3y^2 - 2x^2)^3 - (3x^2 - 2y^2)^3$.

156. Applications of the Remainder Theorem.

The Remainder Theorem (Art. 135) offers a valuable method for determining whether or not an expression contains a binomial factor. By its aid it is easy to ascertain whether a certain binomial is or is not a factor without the labor of division, as will be seen in the examples which follow.

1. Factor the expression $x^3 - 2x - 1$.

Because the expression ends in -1, we know at once that either $x - 1$ or $x + 1$ is a factor, if it has a binomial factor. Putting 1 in place of x in the expression, we have

$$1 - 2 - 1, \text{ which is } \textit{not } 0.$$

Hence, $x - 1$ is not a factor. Putting -1 for x, we have

$$-1 + 2 - 1 = 0.$$

Hence, $x + 1$ is a factor, and the other factor is found by division to be $x^2 - x - 1$.

$\therefore x^3 - 2x - 1 \equiv (x+1)(x^2 - x - 1)$.

CHECK. Let $x = 1$; then $-2 = 2 \times (-1)$.

2. Factor the expression $x^3 - 2x^2 - 2x - 3$.

Because the expression ends in -3, we naturally conclude that $x + 1$, $x + 3$, $x - 1$, or $x - 3$ is probably a factor, the exact divisors of -3 being $1, -1, 3, -3$.

Putting -1 for x, we have

$$-1 - 2 + 2 - 3 \neq 0; \text{ hence } x + 1 \text{ is not a factor.}$$

Similarly, we find that neither $x + 3$ nor $x - 1$ is a factor.

Putting 3 for x, we have

$$27 - 18 - 6 - 3 = 0; \text{ hence } x - 3 \text{ is a factor.}$$

By division, the other factor is found to be $x^2 + x + 1$.

The factors of which term only need to be considered in seeking values to be put in place of x?

Factor the following expressions, using method of Remainder Theorem:

3. $x^3 - 8x + 3$.
4. $x^3 - 3x + 2$.
5. $x^3 - 3x^2 + 4$.
6. $x^2 + 8x + 7$.
7. $a^3 - 7a + 6$.
8. $3x^3 - 4x^2 - 6x + 4$.
9. $x^3 + 3x^2 - 13x - 15$.
10. $x^3 + 8x^2 + 17x + 10$.
11. $m^3 + 3m^2 - 5m - 15$.
12. $x^3 - 4x^2 + 7x - 12$.
13. $x^3 - 39x - 70$.
14. $m^3 - 21m + 20$.
15. $a^3 + 4a^2 + 4a + 3$.
16. $x^3 + 2x^2 - 13x + 10$.
17. $x^4 - 3x^2 + 2x - 8$.
18. $x^4 - x^3 - 7x^2 + x + 6$.
19. $3x^3 + x^2 - 22x - 24$.
20. $x^4 - x^3 - 3x^2 - 8x - 3$.
21. $y^4 - 15y^2 + 10y + 24$.
22. $n^5 - 4n^4 + 19n^2 - 28n + 12$.

157. Algebraic expressions may often appear to have more than one distinct set of factors. For example,

$$(3a - b)(3a - b) \equiv 9a^2 - 6ab + b^2,$$

and

$$(b - 3a)(b - 3a) \equiv 9a^2 - 6ab + b^2.$$

Here the second pair of factors is the same as the first, except that the signs of the terms are changed. In either form the absolute value is the same. If each factor of the first pair is multiplied by -1, the second pair is obtained, but this is equivalent to multiplying the given expression by -1 *twice* (*i.e.*, by $+1$), which does not change it.

It follows that *the signs of any even number of factors may be changed without changing the product.*

158. The following general directions will be found of value in attacking expressions to be factored:

1. *As a first step remove all monomial factors.*

2. *If the expression is in the form of any of the types already considered, factor by applying the given type form. If the expres-*

sion is not in the required form, see if it can be brought under any of the given types, especially in cases of binomials and trinomials of the second degree.

3. *If the expression cannot be brought under any of the given types, the method of the Remainder Theorem should be tried to test for binomial factors.*

MISCELLANEOUS EXAMPLES

159. Factor the following expressions:

1. $x^2 - y^2$.
2. $m^5 + n^5$.
3. $m^4 - 16$.
4. $a^3 + 1$.
5. $x^8 - y^8$.
6. $a^9 - x^9$.
7. $a^4 - 4b^4$.
8. $x^5 - x$.
9. $a^9 - x^6$.
10. $a^{12} - b^8$.
11. $a^6 - 64b^6$.
12. $100x^4 - 36y^2$.
13. $a^4 - 7a^2 + 12$.
14. $m^6 + 14m^3 + 49$.
15. $x^2 - 4x - 21$.
16. $8a^2 + 2a - 3$.
17. $a^8 - 2a^4 - 24$.
18. $a^2 - 10a + 24$.
19. $1 - (x - y)^3$.
20. $a^4 - (b - c)^4$.
21. $x^3 - x^4 - x^2 + x$.
22. $a^3 + 3a^2 + a - 2$.
23. $a^3 - 5a^2 - a + 5$.
24. $x^{10} + 2x^5 + 1$.
25. $x^4 - 8x^2 + 7$.
26. $x^2 - 5x - 14$.
27. $x^2 + 4x - 45$.
28. $cx^2 + 5cx - 84c$.
29. $(2a - 3b)^3 + (3b)^3$.
30. $x^2 - x - z^2 + z$.
31. $a^4 + b^8 + a^2b^4$.
32. $6b^2 - 19b + 15$.
33. $a^2 + b^2 - c^2 + 2ab$.
34. $x^3 - x^2 + 3x + 5$.
35. $(x + y)^6 - 1$.
36. $(a^2 - b^2) - (a^2 - ab)^2$.
37. $10p^2 - 47p + 42$.
38. $(a + b - 3c)^2 - 9c^2$.
39. $63x^2 - 24x - 15$.
40. $a^8 - 7a^4b^4 + b^8$.
41. $4 - 33a^2 + 4a^4$.
42. $a^3 - 6a^2 - a + 30$.
43. $16p^4 + 81q^4 + 36p^2q^2$.
44. $x^3 - 7x^2 + 10x + 6$.
45. $ac + bd - ad - bc$.
46. $x^3 + x^2 + x + 1$.
47. $a^4 + a^2c^2 + c^4$.
48. $m^3 - n^3 + m - n$.
49. $m^4 - 1 - n^2 + 2n$.
50. $a^4 + a^3 - a - 1$.

51. $3mx - nx - 3my + ny$.
52. $4x^2y^2 - (x^2 + y^2 - z^2)^2$.
53. $3x^4 - 6x^2y^3 + 3y^6$.
54. $ax^2 + a - bx^2 - b$.
55. $x^5 + x^4 - 9x - 9$.
56. $x^2 - x^4 + 1 - x^2$.
57. $x^2y^2z^2 - xyz - 30$.
58. $a^2b^2 + 23abc + 90c^2$.
59. $a^6 - 17a^3bc^2 + 72b^2c^4$.
60. $a^4 - 7a^2b^2 + b^4$.
61. $a^4b - a^2b^3 - a^3b^2 + ab^4$.
62. $4x^{4m} - 4x^{2m}y^n + y^{2n}$.
63. $36x^4 - 28x^2y^2 + 4y^4$.
64. $a^4b^6 + 20a^2b^3c + 96c^2$.
65. $x^3 + x^2y + xy^2 + y^3$.
66. $(m - n)x + (n - m)y$.
67. $y^2 - (2a + 3b)y + 6ab$.
68. $x^4 - (a^2 + b^2)x^2 + a^2b^2$.
69. $60 + x^3 - 17x - 4x^2$.
70. $16x^4 - 72x^2y^2 + 81y^4$.
71. $8a^3 - 36a^2b + 54ab^2 - 9b^3$.
72. $4 - 4a + 2mn - m^2 + a^2 - n^2$.
73. $x^2 + 7(m + n)x + 10(m + n)^2$.
74. $8m^2n^5y^5 - 12mn^4y^7 + 4m^2n^3y^8$.
75. $(x + y)^2 - 14(x + y) + 49$.
76. $y^4 - (m^2 + n^2)y^2 + m^2n^2$.
77. $a^3 - 9a^2b + 27ab^2 - 27b^3$.
78. $x^2 - y^2 - z^2 + 2yz + x + y - z$.
79. $mx - nx + ny + py - px - my$.
80. $21x - 5y + 3xy - 35 - 14z - 2yz$.
81. $21(x + y + z)^2 - 17(x + y + z) - 30$.
82. $16x^2 - 8xy + y^2 - z^2 - 10zw - 25w^2$.
83. $-121m^4x^2 + 220m^2n^2xy - 100n^4y^2$.

SOLUTION OF EQUATIONS BY FACTORING

160. Factoring finds a simple and valuable application in the practical solution of many equations of higher degree than the first. The use of the factoring method in solving equations is shown in the examples considered below.

161. As we have seen (Art. 30), to solve an equation is to find the value of the unknown number — the root of the equation — which when substituted in the original equation converts it into an identity.

Thus, 4 is such a value of x in the equation $2x + 3 = 11$. Substituting 4 for x, $2 \times 4 + 3 = 11$.

1. Find the value of x in the equation $x^2 + 3x = 10$.

$x^2 + 3x - 10 = 0$, by subtracting 10.

$(x - 2)(x + 5) = 0$, by factoring.

The first member evidently reduces to 0, if either factor is 0, because the product will then be 0 (Art. 93).

Putting $x - 2 = 0$, then $x = 2$, by adding 2.

Putting $x + 5 = 0$, then $x = -5$, by subtracting 5.

$\therefore x = 2$ or -5; for if $x = 2$, then $2 \times 2 + 3 \times 2 - 10 = 0$; and if $x = -5$, then $-5 \times -5 + 3 \times -5 - 10 = 0$.

2. Find the value of x in the equation $x^2 - 9 = 0$.

$x^2 - 9 = 0$.

$(x + 3)(x - 3) = 0$, by factoring.

The first member evidently reduces to 0, if either factor is 0. Why?

Putting $x + 3 = 0$, then $x = -3$. (Why?)

Putting $x - 3 = 0$, then $x = +3$. (Why?)

$\therefore x = +3$ or -3.

3. Solve the equation $x^3 - 2x^2 - 5x + 6 = 0$.

$x^3 - 2x^2 - 5x + 6 = 0$.

$(x - 1)(x^2 - x - 6) = 0$, by method of Rem. Theorem.

$(x - 1)(x + 2)(x - 3) = 0$.

The first member of this equation will evidently become equal to the second (*i.e.*, the equation will reduce to an identity), if *any* factor of the first member equals 0. Why?

Putting $x - 1 = 0$, then $x = 1$.

Putting $x + 2 = 0$, then $x = -2$.

Putting $x - 3 = 0$, then $x = 3$.

$\therefore x = 1$, or -2, or 3.

Solve the following equations:

4. $x(x-2)=0$.
5. $x^2-1=0$.
6. $x^2-6x=0$.
7. $x^2+3x=0$.
8. $x(x-1)=0$.
9. $x^2=16$.
10. $x(2x+1)=0$.
11. $x^2+4x+4=0$.
12. $x^2-5x+6=0$.
13. $3x^2+2x-5=0$.
14. $x^2-x-2=0$.
15. $x^2-5x-36=0$.
16. $x^2-7x=-12$.
17. $2y^2-y-3=0$.
18. $65-x^2=16$.
19. $6x^2+x-15=0$.
20. $y^2+4y=32$.
21. $x^3-9x^2+27x=27$.
22. $x^3-x^2-x+1=0$.
23. $1-3x+3x^2-x^3=0$.
24. $x^2-3x-40=a^0-1$.
25. $(2x+3)(3x+4)=0$.
26. $x^4-6x^3+11x^2-6x=0$.
27. $x^3-4x^2+10x=3x^2-2x$.

Solve the following problems:

28. If 6 is added to the square of a number, the sum will be equal to 5 times the number. Find the number.

29. If 27 is subtracted from the square of a number, the remainder will be 6 times the number. Find the number.

30. If 20 is added to a certain number, and 5 is subtracted from the same number, the product of the sum and difference thus obtained will be equal to -100. Find the number.

31. The number of square rods in the area of a square field, diminished by 40, is equal to 3 times the number of rods in a side. How many rods in a side of the field?

32. The square of the number expressing the number of dollars a man is worth is greater by 165 than 4 times that number. How much is he worth? What does the negative result mean here?

33. If the square of a number and the number itself be subtracted from the cube of the number, the remainder will be 5 times the number. Find the number (or numbers).

HIGHEST COMMON FACTOR

162. The **Highest Common Factor** of two or more algebraic expressions is the algebraic factor of *highest degree* which is *common* to them. It will of course exactly divide each of them.

Thus, a is a common factor of ab^3 and ab^2c; but the highest common factor is ab^2. The highest common factor of x^2-1 and x^2+3x+2 is $x+1$, which is the only common factor; the factors of x^2-1 being $x+1$ and $x-1$, and those of x^2+3x+2 being $x+1$ and $x+2$.

163. The highest common factor has reference only to the *degree* of the expression and should not be confused with the greatest common divisor in arithmetic. They are not necessarily the same when numerical values are assigned to the letters contained in the expressions. For example, the highest common factor of a^2 and ab^2 is a; but if $a=2$ and $b=4$, the greatest common divisor of 2^2 and 2×4^2 is 4, and not a, or 2.

NOTE. Since common numerical factors have nothing to do with the algebraic divisibility of the expressions and do not affect the *degree* of the highest common factor, it is not necessary to consider them so far as the algebraic highest common factor is concerned. It is customary, however, in many text-books to write the greatest common divisor of the numerical factors as a coefficient of the algebraic highest common factor.

164. By Factoring Method.

When the given expressions can be easily factored, their highest common factor is usually found by inspection. It is necessary merely to factor the expressions and then take the product of all the common factors, giving to each the lowest exponent with which it occurs in any of the expressions.

1. Find the highest common factor of $2\,a^2b^3$, $4\,a^4b^4$, $6\,a^3b^5$.

The literal factors are a and b. The lowest exponent of a in any of the expressions is 2, while that of b is 3. Hence the algebraic H. C. F. is a^2b^3; prefixing as a coefficient the G. C. D. of the numerical factors, we have the result $2\,a^2b^3$.

2. Find the H. C. F. of $(x+y)^2$ and x^2-y^2.

$$(x+y)^2 \equiv (x+y)(x+y).$$
$$x^2-y^2 \equiv (x+y)(x-y).$$

$\therefore$ the highest common factor is $x+y$.

3. What is the H. C. F. of $6\,a^2x^2-24\,a^2$, $2\,ax^2+2\,ax-12\,a$, and $4\,a^2x^2-20\,a^2x+24\,a^2$?

$$6\,a^2x^2-24\,a^2 \equiv 6\,a^2(x^2-4)$$
$$\equiv 6\,a^2(x-2)(x+2).$$
$$2\,ax^2+2\,ax-12\,a \equiv 2\,a(x^2+x-6)$$
$$\equiv 2\,a(x-2)(x+3).$$
$$4\,a^2x^2-20\,a^2x+24\,a^2 \equiv 4\,a^2(x^2-5\,x+6)$$
$$\equiv 4\,a^2(x-2)(x-3).$$

Therefore the H. C. F. is $2\,a(x-2)$.

4. Find the H. C. F. of $a^2-2\,ab+b^2$ and b^2-a^2.

$$a^2-2\,ab+b^2 \equiv (a-b)(a-b) \equiv (b-a)(b-a). \quad \text{(Prove this.)}$$
$$b^2-a^2 \equiv (b+a)(b-a) \equiv -(a-b)(a+b).$$

Here $b-a$ is the highest common factor. But $a-b$ is also the highest common factor. This does not mean that there are two distinct factors, or that $b-a$ is identical with $a-b$, although they are the same in absolute value. If, for example, $b-a$ is an exact divisor of an expression, $-(b-a)$, or $a-b$, is also an exact divisor, but the signs of the quotient will be changed.

Thus, $(b^2-a^2) \div (b-a) \equiv b+a$;

and $(b^2-a^2) \div (a-b) \equiv -b-a$, or $-(b+a)$.

In general, if $m \pm n$ is a factor of any expression, $-(m \pm n)$ is also a factor. It is not customary, however, to state both forms.

5. Find the H. C. F. of x^2-1 and $x^3-4\,x+3$.

$$x^2-1 \equiv (x+1)(x-1).$$

By the Remainder Theorem, $x-1$ is found to be a factor of $x^3-4\,x+3$, while $x+1$ is not. Therefore the H. C. F. is $x-1$.

If the factors of one expression are known, and it is difficult to discover the factors of the others, we can readily ascertain by division or by the Remainder Theorem whether or not the known factors are factors of the other expressions.

Find the H.C.F. of the following sets of expressions:

6. $5\,abx^3,\ 7\,ax^2y,\ abx^3y^2.$
7. $x^3y^2,\ x^2y^3,\ x^4y^4z.$
8. $a^2b^3c^4,\ a^3b^3c^3,\ a^4b^3c^2.$
9. $a^3x^2y,\ a^2x^3y^4,\ a^3xy^2z.$
10. $3\,m^2x,\ 6\,mx^3,\ 9\,m^3x^2.$
11. $6\,bx^n,\ 2\,b^2x^{n+2},\ 4\,bx^{n+1}.$
12. $a(a+b),\ a^2-b^2.$
13. $m^4-n^4,\ (m-n)^4.$
14. $x^2-2\,x+1,\ x^4-1.$
15. $a(x+y)^2,\ b(x+y)^3.$
16. $x^2-x-20,\ x^2-9\,x+20.$
17. $x^2-7\,x+10,\ x^2-12\,x+35.$
18. $x^3-xy^2,\ x^3-2\,x^2y+xy^2.$
19. $x^2-1,\ x^3+1,\ x^2-2\,x-3.$
20. $z^2-2\,z-24,\ z^2+9\,z+20.$
21. $2\,x^2-x-1,\ 3\,x^2-x-2.$
22. $a^9-b^9,\ a^6-b^6,\ a^3-b^3.$
23. $x^2-2\,x,\ x^2-4,\ x^2-4\,x+4.$
24. $x^2-2\,x+1,\ x^2+x-2,\ x^2-1.$
25. $x^4-3\,x^3,\ x^3-9\,x,\ x^2-6\,x+9.$
26. $x^2+(a+b)x+ab,\ x^2+(b+c)x+bc.$
27. $ax^2+2\,a^2x+a^3,\ ax^2-2\,a^2x-3\,a^3.$
28. $x^2-7\,x+10,\ x^3-6\,x^2+11\,x-6.$
29. $m^3+m^2+m-3,\ m^3+3\,m^2+5\,m+3.$
30. $x^4+xy^3,\ xy^3+y^4,\ x^4+x^2y^2+y^4.$
31. $m^2-1,\ m^3-1,\ (m-1)^2(m+1).$
32. $a^2-1,\ a^4-1,\ a^3-2\,a^2+a.$
33. $x^3-a^2x,\ x^2+ax,\ x^4-a^4,\ x^8-a^8.$
34. $a^4+a^2b^2+b^4,\ a^2+ab+b^2.$
35. $ac(a-b)(a-c),\ bc(b-a)(b-c).$
36. $x^2-1,\ x^3-1,\ 2\,x^3-x^2-x-3.$
37. $(x-y)(y-2),\ (2-x)(x^2-y^2).$

165. By Division Method.

When the given expressions cannot be easily factored, their highest common factor can be obtained by a method analogous

to that employed in arithmetic for finding the greatest common divisor of two or more numbers.

This method depends upon two principles:

I. **A factor of an algebraic expression is a factor of any of its multiples.**

This principle has been made familiar in arithmetic. For example, 7, which is a factor of 28, is also a factor of any multiple of 28, as of $2(7 \times 4)$, $3(7 \times 4)$, etc., since to take 28 any number of times does not take out the factor 7.

And, in general, if a is any given algebraic expression of which f and p are the factors, p being the quotient when a is divided by f, then

$$a \equiv fp.$$

$$\therefore am \equiv fpm,$$ by multiplying by any integral expression, m.

That is, if f is a factor of a, being contained p times in a, it is a factor of any multiple of a, as am, being contained in m times the expression pm times.

II. **Any common factor of two algebraic expressions is a factor of their sum, of their difference, and of the sum or the difference of any multiples of the expressions.**

This principle is also seen to be true for arithmetical numbers. For example, 7, which is a common factor of 35 and 126, is also a factor of the sum and of the difference of any multiples of 35 and 126, as of $2(35 + 126)$ and $2(126 - 35)$, etc.

In general, if a and b represent any two algebraic expressions which have a common factor, f, and if p denotes the quotient when a is divided by f, and q the quotient when b is divided by f; then

$$a \equiv fp \quad \text{and} \quad b \equiv fq.$$

$$\therefore am \equiv fpm \text{ and } bn \equiv fqn. \qquad \text{(Why?)}$$

$$\therefore am - bn \equiv fpm - fqn, \text{ by subtracting equals from equals,}$$

$$\equiv f(pm - qn), \text{ by factoring.}$$

That is, if f is a common factor of a and b, being contained in the former p times and in the latter q times, then it is also a factor of the sum and of the difference of any multiples of a and b, as of am and bn, being contained in the former pm times and in the latter qn times; hence in their sum $pm + qn$ times and in their difference $pm - qn$ times.

166. This method is best understood by considering an example. Let it be required to find the highest common factor of $x^3 + x^2 - 2$ and $x^3 + 2x^2 - 3$.

$$
\begin{array}{l}
x^3 + x^2 - 2)x^3 + 2x^2 - 3(1 \\
\quad \underline{x^3 + \ \ x^2 - 2} \\
\qquad x^2 - 1)x^3 + x^2 - 2(x + 1 \\
\qquad\quad \underline{x^3 - x} \\
\qquad\qquad x^2 + x - 2 \\
\qquad\qquad \underline{x^2 \quad - 1} \\
\qquad\qquad\quad x - 1)x^2 - 1(x + 1 \\
\qquad\qquad\qquad\quad \underline{x^2 - x} \\
\qquad\qquad\qquad\qquad x - 1 \\
\qquad\qquad\qquad\qquad \underline{x - 1}
\end{array}
$$

The highest common factor of the given expressions is also a factor of $x^2 - 1$ (their difference), by Prin. II.

And since it is a factor of $x^2 - 1$ and $x^3 + x^2 - 2$, it is also a factor of the second remainder, $x - 1$, by the same principle.

Therefore the highest common factor cannot be of a higher degree than $x - 1$, and it is $x - 1$ if that is a factor of $x^2 - 1$, $x^3 + x^2 - 2$, and $x^3 + 2x^2 - 3$.

But $x - 1$ is a factor of $x^2 - 1$, by trial.

And since it is a factor of itself and of $x^2 - 1$, it is a factor of $x^3 + x^2 - 2$, by Prin. II. $[x^3 + x^2 - 2 \equiv (x + 1)(x^2 - 1) + 1 \times (x - 1)$, the sum of multiples of $x^2 - 1$ and $x - 1$.]

It is also a factor of $x^3 + 2x^2 - 3$, by the same principle.

$\therefore$ $x - 1$ is the H. C. F. of the two given expressions.

167. The **highest common factor of three expressions** may be obtained by finding the highest common factor of two of them, then of this result and the third expression. For the H. C. F.

of the three expressions is a common factor of any two, and hence of their highest common factor (which contains all their common factors), and evidently it cannot be of a higher degree than that of the H. C. F. of any two of them. This method may be extended to any number of expressions.

1. Find the highest common factor of $2ax^3 - ax^2 - 2ax + a$ and $7abx^3 - 10abx^2 - 7abx + 10ab$.

Reserve the common factor a as a factor of the H. C. F. found by division. Reject the factor b, which is not common.

$$
\begin{array}{r}
7x^3 - 10x^2 - 7x + 10 \\
\hline
2x^3 - x^2 - 2x + 1)\overline{14x^3 - 20x^2 - 14x + 20}(7 \\
\underline{14x^3 - 7x^2 - 14x + 7} \\
-13)\overline{-13x^2 + 13} \\
x^2 - 1)\overline{2x^3 - x^2 - 2x + 1}(2x - 1 \\
\underline{2x^3 - 2x} \\
-x^2 + 1 \\
\underline{-x^2 + 1}
\end{array}
$$

$\therefore$ the H. C. F. is $a(x^2 - 1)$.

(*a*) A common monomial factor of the expressions, as the factor a in the preceding, should be removed before the division is made, but it must be made a factor of the H. C. F.

(*b*) If either expression contains a factor not found in the other, it may be rejected, since only *common* factors enter into the H. C. F. Dividing by -13 in the preceding solution rejects the factor -13, which is not common.

(*c*) To avoid fractional quotients it is frequently necessary to introduce numerical factors, as 2 above. Neither this nor the rejection of factors not common affects the *degree* of the H. C. F. The numerical factor introduced into either expression should not already be a factor of the other expression, for it would then be a *common* factor.

Detached coefficients can often be used to advantage in making the divisions.

Find the highest common factor of the following sets of expressions:

2. $2z^2 + z - 3,\ 4z^3 + 8z^2 - z - 6.$

3. $2x^2 - 5x + 2,\ 4x^3 + 12x^2 - x - 3.$

4. $x^4 - 2x^3 + x$, $x^4 - x^3 - x - 1$.

5. $x^3 + 4x^2 + 5x + 2$, $x^2 + 5x + 4$.

6. $x^2 - 4x + 3$, $4x^3 - 9x^2 - 15x + 18$.

7. $6x^3 - 7x^2 - 22x + 32$, $3x^2 + x - 10$.

8. $2x^3 - x^2 - 5x + 3$, $2x^3 - x^2 - x - 3$.

9. $2a^3 + 4a^2 + 2a + 4$, $a^4 - 4a^2 - a - 2$.

10. $3x^3 - 7x^2 + 4x - 4$, $2x^3 - 3x^2 + 2x - 8$.

11. $x^4 - x^3 + 2x^2 - x - 1$, $2x^4 - 2x^2 + x - 1$.

12. $1 - 2a - 5a^2 + 6a^3$, $1 + 5a + 2a^2 - 8a^3$.

13. $12x^4 - x^3 - 30x^2 - 16x$, $6x^4 - 2x^3 - 13x^2 - 6x$.

14. $6x^2 - 13x - 5$, $18x^3 - 51x^2 + 13x + 5$.

15. $x^3 + 2x^2 + 2x + 1$, $x^3 - 4x^2 - 4x - 5$.

16. $x^5 - 2x^4 - 2x^3 + 8x^2 - 7x + 2$, $x^4 - 4x + 3$.

17. $x^4 - x^3 + 2x^2 - x + 1$, $x^4 + x^3 + 2x^2 + x + 1$.

18. $2x^4 + x^3 + 4x - 3$, $3x^4 + 2x^3 - 2x^2 + 3x - 2$.

19. $y^2 + y - 6$, $y^3 - 2y^2 - y + 2$, $y^3 + 3y^2 - 6y - 8$.

20. $x^4 - 10x^3 + 35x^2 - 50x + 24$, $x^3 - 8x^2 + 17x - 10$.

21. $10amx^2 - 7amx - 33am$, $14abx^3 + 11abx^2 - 15abx$.

22. $abx^3 + abx^2 - 9abx - 9ab$, $ax^4 - 7ax^2 - 6ax$.

23. $x^4 - 2x^3y + 2xy^3 - y^4$, $x^4 - 2x^3y + 2x^2y^2 - 2xy^3 + y^4$.

24. $x^3 - x^2 - 5x - 3$, $x^3 + 2x^2 - x - 2$, $x^3 - 2x^2 - 2x + 1$.

25. $2x^2 + 3x + 1$, $2x^2 + 5x + 2$, $2x^3 + 5x^2 - 4x - 3$.

26. $x^2 + x - 12$, $x^2 - 10x + 21$, $x^3 - 6x^2 - 19x + 84$.

27. $6x^4 + x^3 - x$, $4x^3 - 6x^2 - 4x + 3$, $2x^3 + x^2 + x - 1$.

28. $2x^3 + 5x^2 + 5x + 6$, $3x^3 + 5x^2 - 5x - 6$, $x^3 + 4x^2 + 5x + 2$.

29. $x^2 - 11x + 30$, $x^3 - 12x^2 + 41x - 30$, $x^4 - 12x^3 + 47x^2 - 72x + 36$.

LOWEST COMMON MULTIPLE

168. The **Lowest Common Multiple** of two or more algebraic expressions is the expression of *lowest degree* that is exactly divisible by each of them.

Thus, $a^2b^2c^2$ is the lowest common multiple of ab^2c, abc^2, and a^2bc; that of $(a+b)^2$ and a^2-b^2 is $(a+b)^2(a-b)$.

It will be observed that the exponent of each factor in the L. C. M. is the highest exponent that factor has in any of the given expressions. When the expressions contain numerical factors, the L. C. M. of these factors can be found as in arithmetic, and written as a *coefficient* of the algebraic lowest common multiple.

169. By Factoring Method.

When the given expressions can be easily factored, their lowest common multiple is usually found by inspection. It is necessary merely to take the product of all the common factors, giving to each the highest exponent it has in any of the given expressions, and to multiply this product by the L. C. M. of the numerical factors when there are any.

1. Find the lowest common multiple of a^3b^2c, $a^2b^3d^4$, and a^4d^5.

Each letter must appear in the L. C. M. with the highest exponent it has in any of the given expressions.

$\therefore$ $a^4b^3cd^5$ is the lowest common multiple.

2. Find the L. C. M. of a^2-b^2, $a^2+2ab+b^2$, and $a^2-2ab+b^2$.

$$a^2-b^2 \equiv (a+b)(a-b).$$
$$a^2+2ab+b^2 \equiv (a+b)(a+b).$$
$$a^2-2ab+b^2 \equiv (a-b)(a-b).$$
$$\therefore (a+b)^2(a-b)^2 \text{ is the L. C. M.}$$

Find the L. C. M. of the following sets of expressions:

3. a^3b^2, a^2b^2c.
4. $3ax^2$, $4by^2$, $2abx$.
5. n^2, $n(n+mx)$.
6. ab^2, a^3bc^3, ab^2c^4.
7. a^2bx^2, $ab^2x^3y^2$.
8. $3x^3y^2z$, $5x^2y^3z^3$.
9. $an(n+1)$, n^2-1.
10. m^2-n^2, $mn-n^2$, $mn+n^2$.
11. $x+1$, x^2-1, $(x+1)^2$.
12. x^2-16, x^2+x-12.
13. $1-a$, $(1-a)^2$, $(1-a)^3$.
14. $a+1$, $1-a$, a^2+a+1.
15. x^2+6x+8, x^2+4x+4.
16. $(a+2b)^2$, $(a-2b)^2$, a^2-4b^2.
17. $x(x-y)$, $xy^2(x^2-y^2)$, $x^2y(x+y)$.
18. $x^2-2xy+y^2$, x^2-y^2, x^3+y^3.
19. x^2-x-6, x^2+x-2, x^2-4x+3.
20. $x^2+y^2-z^2-2xy$, $x^2-y^2-z^2-2yz$.
21. $1-x+x^2$, $1+x+x^2$, $1+x^2+x^4$.
22. $a(x^2-1)$, $ab(x^2-ax)$, $a^3b(x^3-1)$.
23. $(a+c)^2-b^2$, $(a+b)^2-c^2$, $(b+c)^2-a^2$.
24. $x^4+5x^2y^2+9y^4$, $x^4-2x^3y+x^2y^2-9y^4$.
25. x^2y-xy^2, $3xy(x-y)^2$, $4yz(x-y)^3$.
26. x^2+xy+y^2, x^2-xy+y^2, $x^4+x^2y^2+y^4$.
27. $a-1$, a^2-1, a^3-1, a^4-1, a^5-1.
28. a^2-b^2, a^3-b^3, a^4-b^4, a^5-b^5, a^6-b^6.
29. x^4-y^4, x^3-y^3, x^2-y^2, $x^2-2xy+y^2$, $x^2+2xy+y^2$.

170. By Means of Highest Common Factor.

The highest common factor of two expressions contains *all their common factors;* hence, if it is removed from the expressions, the quotients thus obtained will contain *all the factors not common.* Consequently the product of these quotients and the H. C. F. will be the lowest common multiple.

Thus, the factors common to a^2bx and ab^2y are a and b, and their H. C. F. is ab. Removing this factor, we obtain the quotients ax and b^2y. Hence $ab \times ax \times b^2y$, **or** a^2b^2xy, is the L. C. M.

Let x and y be any two algebraic expressions, f their highest common factor, a and b the quotients when x and y respectively are divided by f; so that

$$x \equiv fa,$$
$$y \equiv fb.$$

Since f, the H. C. F. of x and y, contains all their common factors, a and b can have no common factors; hence their L. C. M. must be fba. That is, the H. C. F. of two expressions multiplied by their other factors is their L. C. M.

But fba is equal to y multiplied by a; hence the L. C. M. of two expressions may be obtained by dividing *either* of them by their H. C. F. and multiplying the quotient by the other.

Thus, the H. C. F. of ax^2y and bxy^2 is xy; $ax^2y \div xy = ax$. Then $ax \times bxy^2 = abx^2y^2$, the L. C. M.

1. Find the lowest common multiple of $x^3 - 6x^2 + 11x - 6$ and $x^3 - 9x^2 + 26x - 24$.

$$\begin{array}{r|l}
x^3 - 6x^2 + 11x - 6 & x^3 - 9x^2 + 26x - 24 \;(1 \\
\quad x^3 - 6x^2 + 11x - 6 & \\
\hline
-3) \quad -3x^2 + 15x - 18 & \\
\hline
\end{array}$$

$$\begin{array}{r|l}
x^2 - 5x + 6 & x^3 - 6x^2 + 11x - 6 \;(x - 1 \\
\quad x^3 - 5x^2 + 6x & \\
\hline
\quad -x^2 + 5x - 6 & \\
\quad -x^2 + 5x - 6 & \\
\hline
\end{array}$$

$\therefore (x^3 - 9x^2 + 26x - 24)(x - 1)$ is the lowest common multiple.

It will be observed that the H. C. F. is $x^2 - 5x + 6$, and that $x - 1$ is the quotient obtained by dividing one expression by the H. C. F.

The lowest common multiple of three or more expressions is obtained by finding the L. C. M. of two of them; then the L. C. M. of this result and a third expression, and so on.

Find the L. C. M. of the following sets of expressions:

2. $a^2 - y^2$, $ay - y^2$.
3. $x^2 - 1$, $x^2 + 2x + 1$.
4. $x^3 + x^2 - 2$, $x^3 + 2x^2 - 3$.
5. $a^3 - 1$, $a^3 - a$.
6. $x^3 + 1$, $x^2 - 1$.
7. $x^2 + x - 2$, $x^2 - 2x - 3$.

8. $m^3-3m+2,\ m^3+2m^2-m-2.$

9. $6x^2-17x+12,\ 12x^2-4x-21.$

10. $3z^3-2z^2-1,\ 4z^3-5z+1.$

11. $a^2+2a-3,\ a^3-7a^2+6a.$

12. $a^3+4a^2+3a-2,\ a^3+a^2-3a+1.$

13. $a^3-a^2+a+3,\ a^3-a+6.$

14. $4a^3-3a^2-8a-1,\ a^4-a^3-4a^2-a+1.$

15. $6x^2-x-1,\ 2x^2+3x-2.$

16. $a^4-a^3+8a-8,\ a^3+4a^2-8a+24.$

17. $x^3-1,\ 2x^2-x-1,\ 3x^2-x-2.$

18. $a^3-2a-1,\ a^3+2a^2+2a+1.$

19. $x^2+7x+9,\ x^2-3x+7,\ x^2-2x+11.$

20. $n^2-1,\ n^3+n^2+n+1,\ n^3-n^2+n-1.$

21. $x^4-10x^2+9,\ x^4+10x^3+20x^2-10x-21.$

22. $x^3-y^3,\ x^3y-y^4,\ y^2(x-y)^2,\ x^2+xy+y^2.$

23. $2a^4-a^3-2a^2-2a-1,\ 3a^4-4a^3+6a^2-7a-8.$

24. $x^3+6x^2+9x+4,\ x^3+2x^2-11x-12,\ x^3-7x-6.$

25. $2x^2+2x-1,\ 3x^3-4x+1,\ 2x^3-3x+1.$

26. $2x^4-11x^3+3x^2+10x,\ 3x^4-14x^3-6x^2+5x.$

27. $3x^2-13x+12,\ 5x^2-17x+6,\ 5x^2+3x-2.$

28. $x^4-ax^3-a^2x^2-a^3x-2a^4,\ 3x^3-7ax^2+3a^2x-2a^3.$

29. $x^3-5x^2+9x-9,\ x^3-x^2-9x+9,\ x^4-4x^2-12x-9.$

30. $x^3+15x^2+66x+80,\ 2x^2+15x-8,\ 2ax^2+3ax-2a.$

31. $a^3-6a^2+11a-6,\ a^3-9a^2+26a-24,\ a^3-8a^2+19a-12.$

32. Show that the product of the L. C. M. and the H. C. F. of two expressions is equal to the product of the given expressions.

FRACTIONS

171. An **Algebraic Fraction** is an expression of unexecuted division written in the form $\frac{a}{b}$, the dividend being called the *numerator*, the divisor the *denominator*.

In arithmetic the denominator is said to denote the number of *equal* parts into which some whole is divided. Under this definition all denominators must be positive *integers*, and therefore the term "fraction" does not include such expressions as $\frac{5}{\frac{1}{2}}$, $\frac{2}{-3}$, $\frac{2}{\sqrt{3}}$, etc.

In algebra the definition is extended so as to embrace any expression in the fractional *form* $\frac{a}{b}$. Either a or b, or both, may be fractional, positive, negative, etc.

172. By the definition of a fraction, $\frac{a}{b}$ represents a number which, when multiplied by b, equals a.

That is, $$\frac{a}{b} \times b \equiv a.$$

173. The **Sign of a Fraction** is written before the line separating numerator from denominator. It belongs to the fraction as a whole.

Thus, in $\frac{a}{x}$ the sign of the fraction is $+$, the signs of a and x being $+$; but if the sign of either a or x were $-$, the sign of the fraction would be $-$. This follows from the rule of signs in division.

$$+\frac{-a}{x} \equiv +\left(-\frac{a}{x}\right) \equiv -\frac{a}{x}, \quad +\frac{a}{-x} \equiv +\left(-\frac{a}{x}\right) \equiv -\frac{a}{x}.$$

$$-\frac{-a}{x} \equiv -\left(-\frac{a}{x}\right) \equiv +\frac{a}{x}, \quad -\frac{a}{-x} \equiv -\left(-\frac{a}{x}\right) \equiv +\frac{a}{x}.$$

$$\frac{+a}{+b} \equiv +\frac{a}{b}, \quad \frac{-a}{-b} \equiv +\frac{a}{b}, \quad \frac{+a}{-b} \equiv -\frac{a}{b}, \quad \frac{-a}{+b} \equiv -\frac{a}{b}.$$

It is thus seen that *the sign of a fraction may be changed, provided we change the sign of either the numerator or the denominator of the fraction, but not if we change the sign of both.*

When either numerator or denominator is a polynomial, its sign is changed by changing the sign of *each* of its terms.

Change the signs of each of the following fractions:

1. $\frac{-4}{2}$.
2. $\frac{6}{-3}$.
3. $\frac{5}{4}$.
4. $\frac{x}{a}$.
5. $\frac{2}{-x}$.
6. $\frac{-a}{-c}$.
7. $\frac{a-b}{x-y}$.
8. $-\frac{3}{c+d}$.
9. $\frac{-x}{-2c}$.
10. $-\frac{-m-n}{m+2}$.
11. $\frac{3(a-b)}{-(c-a)}$.
12. $-\frac{x+y-a}{-a+x}$.

174. Changing the signs of an *even* number of factors does not change the sign of the product; changing the signs of an *odd* number of factors does change the sign of the product (Art. 157). Therefore:

I. The *signs* of an *even number of factors* of the numerator or of the denominator of a fraction *may be changed without changing the sign of the fraction.*

II. The *signs* of an *odd number of factors* of the numerator or of the denominator of a fraction *may be changed, provided we change the sign of the fraction.*

From the preceding principles show that

1. $\frac{a}{b-a} \equiv \frac{-a}{a-b}$.
2. $\frac{a-b-c}{-c} \equiv \frac{b-a+c}{c}$.
3. $\frac{x}{2(x-y)} \equiv -\frac{x}{2(y-x)}$.
4. $\frac{a+b}{(x-y)(b-c)} \equiv \frac{a+b}{(y-x)(c-b)}$.
5. $\frac{3}{(m-n)(a-b)} \equiv -\frac{3}{(m-n)(b-a)}$.

Is the sign of the numerator changed by changing the signs of two of its factors? Does that change the sign of the fraction?

Is the sign of the denominator changed by changing the signs of three of its factors? Does that change the sign of the fraction?

REDUCTION OF FRACTIONS

175. Reduction to Lowest Denominators.

In both arithmetic and algebra we may cancel common factors from the numerator and denominator, and when they have no factor in common remaining, the fraction is said to be expressed with its *lowest denominator.*

Thus, in the fraction $\frac{a^2x}{ax^2}$ we have the common factor ax; hence

$$\frac{a^2x}{ax^2} \equiv \frac{ax \times a}{ax \times x} \equiv \frac{a}{x}.$$

176. PRINCIPLE OF REDUCTION. **Multiplying or dividing both numerator and denominator of a fraction by the same number,** not 0, **does not change the value of the fraction.**

This principle may be proved as follows:

By definition of a fraction,

$$\frac{a}{b} \times b \equiv a, \ a \text{ and } b \text{ being any integers.}$$

$$\therefore \frac{a}{b} \times b \times m \equiv am, \text{ by multiplying by any number } m.$$

$$\therefore \frac{a}{b} \times bm \equiv am, \text{ by Associative Law.}$$

$$\therefore \frac{a}{b} \equiv \frac{am}{bm}, \text{ by dividing by } bm.$$

That is, the numerator and denominator of the fraction $\frac{a}{b}$ may be multiplied by any number m, or those of the fraction $\frac{am}{bm}$ divided by m, without changing the value of either fraction.

177. Changing the sign of every term in both numerator and denominator does not alter the value of the fraction; for this is equivalent to multiplying (or dividing) both by -1.

Thus, if $a = 3$ and $x = 1$,

$$\frac{a - x}{2x - a} = \frac{3 - 1}{2 - 3} = \frac{2}{-1} = -2.$$

Changing signs, $$\frac{x - a}{a - 2x} = \frac{1 - 3}{3 - 2} = \frac{-2}{1} = -2.$$

178. The common factors should be removed (or canceled) from numerator and denominator by inspection if possible. If this cannot be done, their H. C. F. should be found and canceled.

The student should notice that it is only factors common to every term of both numerator and denominator that can be canceled.

1. Reduce $\frac{x^4m^2n^3}{x^3m^2n^6}$ to lowest denominator.

Canceling the common factors x^3, m^2, n^3, the fraction becomes $\frac{x}{n^3}$; and since there are no other factors in common, the fraction is reduced to its lowest denominator.

CHECK. Let $x = 3$, $m = 2$, $n = 1$; then $\frac{81 \times 4 \times 1}{27 \times 4 \times 1} = \frac{3}{1}$.

2. Reduce $\frac{a^2 - b^2}{a^2 - 2ab + b^2}$ to lowest denominator.

$$\frac{a^2 - b^2}{a^2 - 2ab + b^2} \equiv \frac{(a + b)(a - b)}{(a - b)^2}.$$

When the common factor $(a - b)$ is canceled,

$$\frac{(a + b)(a - b)}{(a - b)^2} \equiv \frac{a + b}{a - b},$$

and the fraction is reduced to its lowest denominator.

CHECK. Let $a = 2$, $b = 1$. Why not $a = b = 1$?

3. Reduce $\frac{x^3 + x^2 - 2}{x^3 + 2x^2 - 3}$ to lowest denominator.

By the division method the H. C. F. is found to be $x - 1$. Canceling this factor,

$$\frac{x^3 + x^2 - 2}{x^3 + 2x^2 - 3} \equiv \frac{x^2 + 2x + 2}{x^2 + 3x + 3}.$$

A factor of both numerator and denominator is a factor of their difference (Art. 165, II). The work can often be shortened if this difference is easily factored. In this case, the factors of the first remainder (the difference), $x^2 - 1$, are $x + 1$ and $x - 1$; of these only $x - 1$ is a factor of $x^3 + x^2 - 2$, as found by trial or by the Factor Law (Art. 136).

Reduce each of the following fractions to lowest denominator:

4. $\frac{a^2bc}{a^3bc^2}$.

5. $\frac{4\,a^3bx^2y^3}{7\,bx^3y^3z^2}$.

6. $\frac{-\,ab^2cd^4}{a^2bc^2d^3}$.

7. $\frac{3\,x^2y^3z^4}{6\,x^3y^2z^5}$.

8. $\frac{2\,(a+b)^2}{6\,(a+b)}$.

9. $\frac{x\,(c^2-d^2)}{a\,(c+d)}$.

10. $\frac{x-z}{3\,x-3\,z}$.

11. $\frac{a^nb^{n+1}}{a^{n+1}b^n}$.

12. $\frac{6\,x^2y+3\,x^2y^2}{3\,x^3y^2+6\,x^3y}$.

13. $\frac{2\,a^3b-6\,a^2b}{a^2-8\,a+15}$.

14. $\frac{2-m}{(m^2-4)n}$.

15. $\frac{ax+bx}{n(a^2+2\,ab+b^2)}$.

16. $\frac{a^2-x^2}{a^3-x^3}$.

17. $\frac{x^4-y^4}{x^6-y^6}$.

18. $\frac{x^3+27}{x^2-2\,x-15}$.

19. $\frac{x^3-y^3}{x^2+xy+y^2}$.

20. $\frac{2\,x^2+3\,x+1}{x^2-x-2}$.

21. $\frac{x^4-16}{(x+2)\,(x-2)}$.

22. $\frac{3\,a^2+23\,a-36}{4\,a^2+33\,a-27}$.

23. $\frac{6\,x^2-xy-2\,y^2}{6\,x^2-7\,xy+2\,y^2}$.

24. $\frac{4\,x^2-8\,x+3}{4\,x^2+4\,x-3}$.

25. $\frac{a^3-8}{a^3-2\,a^2+a-2}$.

26. $\frac{x^3-7\,x+6}{x^3-6\,x^2+11\,x-6}$.

27. $\frac{(x^4-y^4)\,(x^3-y^3)}{(x^6-y^6)\,(x-y)}$.

28. $\frac{(a+b)^2-4\,c^2}{a^2-(b+2\,c)^2}$.

29. $\frac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2}$.

30. $\frac{6\,x^3+x^2-5\,x-2}{6\,x^3-x^2-3\,x-2}$.

31. $\frac{a^3-7\,a^2+16\,a-12}{3\,a^3-14\,a^2+16\,a}$.

32. $\frac{x^2+(a+b)x+ab}{(x+a)\,(x+b)\,(x+c)}$.

33. $\frac{a^3-a^2b-ab^2+b^3}{a^4-a^3b+ab^3-b^4}$.

34. $\frac{x^{2n}+2\,x^n+1}{x^{2n}+3\,x^n+2}$.

35. $\frac{a^2+b^2-c^2+2\,ab}{a^2-b^2+c^2+2\,ac}$.

36. $\frac{(x+y)^4-(x-y)^4}{(x-y)^4-(x+y)^4}$.

179. Fractions reduced to Integral or Mixed Expressions, and the Reverse.

Since a fraction is an expression of unexecuted division, it may be reduced to an integral or mixed form by performing the division indicated, provided the degree of the numerator equals or exceeds that of the denominator.

The quotient is an integral expression if the division is exact; otherwise it is a mixed expression.

Thus, $\frac{a^2+2ab+b^2}{a+b} \equiv a+b$; but $\frac{a^2+2ab}{a+b} \equiv a+\frac{ab}{a+b}$.

180. It is seen from the preceding that the *quotient* is the integral part of the mixed expression, while the *remainder* (if any) is the numerator and the *divisor* the denominator of the fractional part. Hence, to reduce a mixed expression to a fractional form, it is necessary simply to reverse the process of the last article.

That is, *multiply the integral part by the denominator, add* (algebraically) *the numerator to the product, and write the denominator under the result.*

Thus, $a+\frac{ab}{a+b} \equiv \frac{a(a+b)+ab}{a+b} \equiv \frac{a^2+2ab}{a+b}$,

and $a-\frac{a-b}{a+b} \equiv \frac{a(a+b)-(a-b)}{a+b} \equiv \frac{a^2+ab-(a-b)}{a+b}$.

1. Reduce $\frac{x^2-3x-5}{x-1}$ to a mixed expression.

$$\frac{x^2-3x-5}{x-1} \equiv x-2-\frac{7}{x-1}.$$

What is done with the remainder? What sign is placed before the fractional part of the quotient?

2. Express $x+y$ as a fraction with denominator $x-y$.

$$\frac{x+y}{1} \equiv \frac{(x+y)(x-y)}{(x-y)} \quad \text{(Art. 176)}$$
$$\equiv \frac{x^2-y^2}{x-y}.$$

3. Reduce $a+x-\frac{a^2+x^2}{a+x}$ to fractional form

$$a+x-\frac{a^2+x^2}{a+x} \equiv \frac{(a+x)(a+x)-(a^2+x^2)}{a+x}$$
$$\equiv \frac{a^2+2ax+x^2-a^2-x^2}{a+x}$$
$$\equiv \frac{2ax}{a+x}.$$

The bar, or line, between numerator and denominator is a sign of aggregation as well as a sign of division; hence, if the sign of the fraction is $-$, the sign of every term of the numerator must be changed when it is added algebraically, as in the preceding solution.

Reduce to integral or mixed expressions:

4. $\frac{an+m}{n}$.

5. $\frac{x^2+2x-1}{x}$.

6. $\frac{x^3-2x^2}{x^2-x+1}$.

7. $\frac{x^3+1}{x+1}$.

8. $\frac{a^4+b^4-2a^2b^2}{a^2-b^2}$.

9. $\frac{x^4+y^4}{x-y}$.

10. $\frac{ax+2x^2}{a+x}$.

11. $\frac{x^3-y^3}{x+y}$.

12. $\frac{a^4+2a^2+a-1}{a^2+2}$.

13. $\frac{x^3-y^3-1}{x-y}$.

14. $\frac{x^2-2x+5}{x-1}$.

15. $\frac{1-a-ab+a^2b}{ab-b}$.

16. $\frac{a^4+a^2b^2+b^4}{a^2+ab+b^2}$.

17. $\frac{x^{3n}+1}{x^{2n}-1}$.

18. $\frac{x^3-y^3-z^3}{x-y}$.

19. $\frac{a^2+2ab+b^2+c^2}{a+b+c}$.

Reduce to fractional form:

20. $mn+\frac{a-m^2n^2}{mn}$.

21. $x^2+y^2+\frac{1}{x^2+y^2}$.

22. $a-4-\frac{a-16}{a+4}$.

23. $a-\frac{2ac-c^2}{a}$.

24. $a+b-\frac{c+d}{a-b}$.

25. $x+1+\frac{4}{x-3}$.

26. $1-\frac{a^2-ab+b^2}{a^2+b^2}$.

27. $a+3b+\frac{9b^2}{a-3b}$.

28. $c^2+d^2-\frac{ac+bd^2}{a+b}$.

29. $1-\frac{x-y}{x+y}$.

30. $a+x-\frac{x^2}{a-x}$.

31. $x-1-\frac{x-1}{x^2+x+1}$.

32. Write x^2+2x+1 as a fraction with the denominator $x+1$.

33. Change the integral expression $m-n$ to a fractional one having the denominator $m+n$.

181. When two or more fractions have the same denominator, they are said to have a *common* denominator; and when this is the L. C. M. of the denominators, it is called the **Lowest Common Denominator** of the fractions.

Thus, $\frac{a}{x}$ and $\frac{b}{x}$ have the *common* denominator x; it is also the lowest common denominator. The fractions $\frac{a}{a^2-1}$ and $\frac{b}{a+1}$ have different denominators, but we can write them thus: $\frac{a}{a^2-1}$ and $\frac{b(a-1)}{a^2-1}$. Here a^2-1 is the L. C. M. of a^2-1 and $a+1$, and is the lowest common denominator of the given fractions.

182. Reduction to Lowest Common Denominator.

Since the numerator and denominator of a fraction may be multiplied by the same number without changing the value of the fraction (Art. 176), in algebra we can reduce any fractions to identical fractions having the lowest common denominator by the method already learned in arithmetic.

That is, *multiply both numerator and denominator of each fraction by the quotient obtained from dividing their lowest common denominator (which is the* L. C. M. *of the denominators) by the denominator of that fraction.*

The fractions should first be reduced to their lowest denominators.

1. Reduce $\frac{ab}{c^2d}$ and $\frac{-b}{cd^2}$ to identical fractions having the lowest common denominator.

The L. C. M. of c^2d and cd^2 is c^2d^2.

$$c^2d^2 \div c^2d = d, \text{ and } \frac{ab \times d}{c^2d \times d} = \frac{abd}{c^2d^2}.$$

$$c^2d^2 \div cd^2 = c, \text{ and } \frac{-b \times c}{cd^2 \times c} = \frac{-bc}{c^2d^2}.$$

2. Reduce $\frac{a-b}{a+b}$, $\frac{a+b}{a^2-b^2}$, and $\frac{a^2+b^2}{a-b}$ to identical fractions having the lowest common denominator.

The L. C. M. of the denominators is $a^2 - b^2$.

Dividing this by each denominator, we have the quotients $a - b$, 1, and $a + b$, respectively.

Multiplying both numerator and denominator of each fraction by the corresponding quotient, we obtain

$$\frac{(a-b)^2}{a^2-b^2}, \quad \frac{a+b}{a^2-b^2}, \quad \text{and} \quad \frac{(a+b)(a^2+b^2)}{a^2-b^2},$$

the required fractions.

Reduce the following to identical fractions having the lowest common denominator:

3. $\frac{c}{x}, \frac{d}{xy}, \frac{d}{x^2y^2}$.

4. $\frac{1}{2ab}, \frac{2}{4bx}, \frac{3}{2bc}$.

5. $\frac{x}{a^2-1}, \frac{1}{a+1}, \frac{1}{a-1}$.

6. $\frac{a}{x-a}, \frac{x}{x-a}, \frac{a^2}{x^2-a^2}$.

7. $\frac{x-2}{x+2}, \frac{x+2}{x-2}, \frac{x^2+4}{x^2-4}$.

8. $\frac{3}{m}, \frac{2}{2m-1}, \frac{1}{4m^2-1}$.

9. $\frac{a}{3-a}, \frac{a^2}{27-a^3}$.

10. $\frac{1}{1+x}, \frac{2}{1-x^2}, \frac{3}{1-x^3}$.

11. $\frac{a}{1-ax}, \frac{x}{1+ax}, \frac{-ax}{1+ax}$.

12. $\frac{1}{1-a}, \frac{a}{a^2-a}, \frac{2}{a^2-1}$.

13. $\frac{x+2}{x^2-3x-4}, \frac{x-1}{x^2-2x-8}$.

14. $\frac{x+1}{x^2+x-6}, \frac{x-1}{x^2+4x+3}$.

15. $\frac{1}{(m+n)(m-n)}, \frac{1}{(m+n)^2}, \frac{1}{(m-n)^2}$.

16. $\frac{1}{2a(a-b)}, \frac{2}{3b(a^2-b^2)}, \frac{3}{6c(a+b)}$.

17. $\frac{x}{a^2b(x-a)}, \frac{y}{ab^2(x-a)}, \frac{z}{ab(x^2-a^2)}$.

18. $\frac{a}{x^2+2x-8}, \frac{a}{x^2-5x+6}, \frac{2a}{2x^2+x-10}$.

19. $\dfrac{a-b}{a^2-b(a-b)}, \quad \dfrac{a+b}{a^2+b(a+b)}, \quad \dfrac{2a^3b^2}{a^6-b^6}.$

20. $\dfrac{1}{x^2-ax+a^2}, \quad \dfrac{1}{x^2+ax+a^2}, \quad \dfrac{a^2}{x^4+a^2x^2+a^4}.$

ADDITION AND SUBTRACTION

183. In algebra, as in arithmetic, it is necessary that fractions have a common denominator before they can be added or subtracted. From the proof given in Art. 126, it follows that

$$\frac{a}{m}+\frac{b}{m}+\frac{c}{m}\equiv\frac{a+b+c}{m}.$$

Therefore, just as $\dfrac{2}{7}+\dfrac{3}{7}=\dfrac{2+3}{7}$, or $\dfrac{5}{7}$, so $\dfrac{a}{c}+\dfrac{b}{c}\equiv\dfrac{a+b}{c}$.

Hence, to add fractions having a common denominator, *add the numerators and write the result over the common denominator.*

A similar method applies in subtracting fractions having a common denominator.

184. We have already seen that if the sign of each term of both numerator and denominator is changed, the value of the fraction is not altered (Art. 173).

Since the bar separating numerator and denominator serves as a sign of aggregation, it is evident that the same principles apply in changing the sign before the fraction as in changing the sign before expressions inclosed in parentheses. Hence, *we may change the sign of each term of the numerator (or denominator), provided we change the sign of the fraction.*

It has also been shown that the signs of any even number of factors may be changed without altering the product (Art. 174). It follows that *the signs of an even number of factors in numerator, denominator, or both, can be changed without altering the value of the fraction.*

These principles find frequent application in changing fractions to a more convenient form for addition and subtraction.

1. Find the sum of $\frac{x+y}{x-y}$ and $\frac{x-y}{x+y}$.

The L. C. M. of $x-y$ and $x+y$ is x^2-y^2.

$$\therefore \frac{x+y}{x-y} \equiv \frac{(x+y)(x+y)}{(x-y)(x+y)} \equiv \frac{x^2+2xy+y^2}{x^2-y^2},$$

and

$$\frac{x-y}{x+y} \equiv \frac{(x-y)(x-y)}{(x-y)(x+y)} \equiv \frac{x^2-2xy+y^2}{x^2-y^2}.$$

$$\therefore \frac{x+y}{x-y} + \frac{x-y}{x+y} \equiv \frac{(x^2+2xy+y^2)+(x^2-2xy+y^2)}{x^2-y^2}$$

$$\equiv \frac{2(x^2+y^2)}{x^2-y^2}.$$

CHECK. Let $x=2$, $y=1$; then $3+\frac{1}{3}=\frac{10}{3}$.

2. Subtract $\frac{x-y}{x+y}$ from $\frac{x+y}{x-y}$.

$$\frac{x+y}{x-y} - \frac{x-y}{x+y} \equiv \frac{(x^2+2xy+y^2)-(x^2-2xy+y^2)}{x^2-y^2}$$

$$\equiv \frac{x^2+2xy+y^2-x^2+2xy-y^2}{x^2-y^2}$$

$$\equiv \frac{4xy}{x^2-y^2}.$$

Since the fraction is preceded by the sign $-$, the entire numerator is to be subtracted; hence the sign of each term in the numerator of the subtrahend must be changed (Art. 180, Ex. 3).

3. Combine and simplify $\frac{x-1}{x-2} + \frac{x-3}{x^2-4} - \frac{x+1}{x+2}$.

The L. C. M. of the denominators is x^2-4.

$$\frac{x-1}{x-2} \equiv \frac{(x-1)(x+2)}{x^2-4} \equiv \frac{x^2+x-2}{x^2-4}.$$

$$\frac{x+1}{x+2} \equiv \frac{(x+1)(x-2)}{x^2-4} \equiv \frac{x^2-x-2}{x^2-4}.$$

$$\therefore \frac{x-1}{x-2} + \frac{x-3}{x^2-4} - \frac{x+1}{x+2} \equiv \frac{(x^2+x-2)+(x-3)-(x^2-x-2)}{x^2-4}$$

$$\equiv \frac{x^2+x-2+x-3-x^2+x+2}{x^2-4}$$

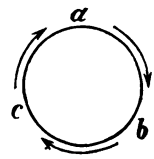
$$\equiv \frac{3(x-1)}{x^2-4}.$$

CHECK. Let $x=3$; then $\frac{2}{1}+\frac{0}{5}-\frac{4}{5}=\frac{6}{5}$.

4. Simplify $\dfrac{1}{(a-b)(a-c)}-\dfrac{1}{(b-a)(b-c)}-\dfrac{1}{(c-a)(c-b)}$.

$$\frac{1}{(a-b)(a-c)}-\frac{1}{(b-a)(b-c)}-\frac{1}{(c-a)(c-b)}$$
$$\equiv -\frac{1}{(c-a)(a-b)}+\frac{1}{(a-b)(b-c)}+\frac{1}{(b-c)(c-a)}$$
$$\equiv \frac{-(b-c)+(c-a)+(a-b)}{(a-b)(b-c)(c-a)} \equiv \frac{-2(b-c)}{(a-b)(b-c)(c-a)}$$
$$\equiv \frac{-2}{(a-b)(c-a)}.$$

This form of solution is given as an example of *cyclic order*; a to b, b to c, and c to a.



Thus we make L. C. M. $(a-b)(b-c)(c-a)$. Show why changes of signs of fractions are admissible.

Use the cyclic order in simplifying Exs. 35, 37, and 39 of this exercise.

Simplify the following expressions:

5. $\dfrac{3+m}{n}+\dfrac{4-am}{an}$.

6. $\dfrac{m}{n}+\dfrac{2m}{3n}-\dfrac{3m}{6n}$.

7. $\dfrac{2}{x^2-4}+\dfrac{3}{x+2}$.

8. $\dfrac{2}{1+a}+\dfrac{a^2+1}{a(1+a)}$.

9. $\dfrac{a}{a-b}+\dfrac{b}{b-a}$.

10. $\dfrac{x+y}{y}-\dfrac{2x}{x+y}-\dfrac{x^2}{y(x+y)}$.

11. $\dfrac{1}{x-m}+\dfrac{1}{x+m}-\dfrac{2}{x}$.

12. $\dfrac{4a^2+b^2}{4a^2-b^2}-\dfrac{2a-b}{2a+b}$.

13. $\dfrac{x^2-x-1}{x^3-1}+\dfrac{1}{x-1}-\dfrac{x}{x-1}$.

14. $\dfrac{a^2-bc}{bc}-\dfrac{ac-b^2}{ca}-\dfrac{ab-c^2}{ab}$.

15. $\dfrac{a+b}{a-b}-\dfrac{a^2+b^2}{b^2-a^2}+\dfrac{b-a}{a+b}$.

16. $\dfrac{1}{a-b}-\dfrac{1}{a+b}$.

17. $\dfrac{2}{3a}-\dfrac{3}{4ax}+\dfrac{1}{x}$.

18. $\dfrac{1}{a}-\dfrac{1}{b}+\dfrac{1}{c}$.

19. $\dfrac{a-b}{a^2b}+\dfrac{a-b}{ab^2}$.

20. $\dfrac{a}{x^2-y^2}+\dfrac{b}{(x+y)^2}-1$.

21. $\dfrac{m-1}{m-2}-\dfrac{m-3}{m-1}+1$.

22. $\dfrac{2-x}{x^2-9}+\dfrac{1}{x+3}-\dfrac{3+x}{(x+3)^2}$.

23. $\dfrac{a+1}{a^2+a+1}-\dfrac{a-1}{a^2-a+1}$.

24. $\dfrac{m}{m-n}+\dfrac{2mn}{n^2-m^2}-\dfrac{2m}{m+n}$.

25. $\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x}{x+y}+\dfrac{y}{y-x}$.

26. $\dfrac{3}{a^n+1}-\dfrac{2}{a^n-1}+\dfrac{4}{a^{2n}-1}$.

27. $\dfrac{a-x}{a-b}+\dfrac{a-b}{a-x}-\dfrac{(a-b)^2}{(a-x)(a-b)}$.

28. $\dfrac{2x+1}{6x^2+x-1}+\dfrac{6x-1}{(3x-1)^2}-\dfrac{1}{3x-1}$.

29. $\dfrac{1}{x^4+x^2+1}-\dfrac{1}{x^2+x+1}+\dfrac{1}{x^2-x+1}$.

30. $\dfrac{2}{x^2-3x+2}+\dfrac{2}{x^2-x-2}-\dfrac{1}{x^2-1}$.

31. $\dfrac{x}{x^2-x-20}+\dfrac{x}{x^2-8x+15}-\dfrac{x}{x^2-9x+20}$.

32. $\dfrac{a}{(b-c)(c-a)}-\dfrac{b}{(a-c)(a-b)}+\dfrac{c}{(b-a)(c-b)}$.

33. $\dfrac{a+b}{a^3-b^3}+\dfrac{a-b}{a^3+b^3}+\dfrac{2(b^2-a^2)}{a^4+a^2b^2+b^4}$.

34. $\dfrac{2x}{x^2-x-20}+\dfrac{3x}{x^2-8x+15}-\dfrac{5x}{x^2-9x+20}$.

35. $\dfrac{x^2}{(x-y)(x-z)}+\dfrac{y^2}{(y-x)(y-z)}+\dfrac{z^2}{(z-x)(z-y)}$.

36. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{(c-a)(a-b)}-\dfrac{a-c}{(a-b)(b-c)}$.

37. $\dfrac{2}{(m-2)(m-3)}-\dfrac{3}{(3-m)(4-m)}-\dfrac{1}{(m-4)(2-m)}$.

38. $\left(\dfrac{x+y}{x-y}+\dfrac{x^2+y^2}{x^2-y^2}\right)-\left(\dfrac{x-y}{x+y}-\dfrac{x^2+y^2}{x^2-y^2}\right)$.

39. $\dfrac{z+x-y}{(x-y)(y-z)}+\dfrac{y+z-x}{(z-x)(x-y)}-\dfrac{x+y+z}{(y-z)(x-z)}$.

40. $\dfrac{x+a}{x^2-(b+c)x+bc}-\dfrac{x+b}{x^2-(a+c)x+ac}+\dfrac{x+c}{x^2-(a+b)x+ab}$.

MULTIPLICATION

185. Multiplication of fractions in algebra is performed as in arithmetic, *by taking the product of the numerators for the numerator of the product, and the product of the denominators for its denominator.*

The principle involved in this method is proved as follows:

If $\frac{a}{b}$ and $\frac{c}{d}$ are two fractions whose product is p, then

$$p \cdot b \cdot d \equiv \frac{a}{b} \times \frac{c}{d} \times b \times d,$$

$$\equiv \frac{a}{b} \times b \times \frac{c}{d} \times d, \text{ by Commutative Law,}$$

$$\equiv a \times c, \text{ by definition of fraction.}$$

Dividing now by bd, $$p \equiv \frac{ac}{bd}.$$

This method can be extended to product of more than two fractions.

186. Powers of Fractions.

Since by multiplication $\frac{a}{b} \times \frac{a}{b}$ equals $\frac{a^2}{b^2}$, and $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b}$ equals $\frac{a^3}{b^3}$, it is evident that the square of $\frac{a}{b}$ is $\frac{a^2}{b^2}$, and the cube of $\frac{a}{b}$ is $\frac{a^3}{b^3}$. In general $\left(\frac{a}{b}\right)^n \equiv \frac{a^n}{b^n}$; for, by definition of exponent,

$$\left(\frac{a}{b}\right)^n \equiv \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} \cdots \text{ to } n \text{ factors},$$

$$\equiv \frac{a \times a \times a \cdots \text{ to } n \text{ factors}}{b \times b \times b \cdots \text{ to } n \text{ factors}} \qquad \text{(Art. 185)}$$

$$\equiv \frac{a^n}{b^n}.$$

That is, *any (integral) power of a fraction is equal to that power of the numerator divided by the like power of the denominator.*

1. Find the product of $\frac{ab^2 - b^3}{a^2 + ab}$ and $\frac{a^3 - ab^2}{2b^2}$.

$$\frac{ab^2 - b^3}{a^2 + ab} \times \frac{a^3 - ab^2}{2b^2} \equiv \frac{b^2(a-b)}{a(a+b)} \times \frac{a(a+b)(a-b)}{2b^2} \equiv \frac{(a-b)^2}{2}.$$

What is the advantage of factoring the expressions? If possible, always factor before multiplying.

2. Find the product of $\frac{ax+x^2}{2b-cx}$, $\frac{2bx-cx^2}{(a+x)^2}$, and $a+x$.

$$\frac{ax+x^2}{2b-cx}\times\frac{2bx-cx^2}{(a+x)^2}\times\frac{a+x}{1}\equiv\frac{x(a+x)}{2b-cx}\times\frac{x(2b-cx)}{(a+x)^2}\times\frac{a+x}{1}$$
$$\equiv x^2.$$

Why may $a+x$ be written $\frac{a+x}{1}$? How may a fraction be multiplied by an integer? Does multiplying the numerator of a fraction by any number multiply the fraction by that number? Show that dividing the denominator has the same effect. Dividing the numerator by any number has what effect on the fraction?

3. Find the product of $\frac{x^2-9}{x^2-x-20}$, $\frac{x^2+x-12}{x^2+x-6}$, and $\frac{x^2-3x-10}{x^2-4}$.

$$\frac{x^2-9}{x^2-x-20}\times\frac{x^2+x-12}{x^2+x-6}\times\frac{x^2-3x-10}{x^2-4}$$
$$\equiv\frac{(x+3)(x-3)}{(x+4)(x-5)}\times\frac{(x+4)(x-3)}{(x+3)(x-2)}\times\frac{(x+2)(x-5)}{(x+2)(x-2)}$$
$$\equiv\frac{(x-3)^2}{(x-2)^2}.$$

4. Simplify $\left(\frac{x^2-a^2}{x-a}\right)^2$.

$$\left(\frac{x^2-a^2}{x-a}\right)^2\equiv\frac{(x^2-a^2)^2}{(x-a)^2}$$
$$\equiv\frac{(x+a)^2(x-a)^2}{(x-a)^2}$$
$$\equiv(x+a)^2.$$

5. Find the product of $\frac{1}{a}+\frac{1}{b}$ and $\frac{1}{a}-\frac{1}{b}$.

$$\left(\frac{1}{a}+\frac{1}{b}\right)\left(\frac{1}{a}-\frac{1}{b}\right)\equiv\left(\frac{1}{a}\right)^2-\left(\frac{1}{b}\right)^2$$
$$\equiv\frac{1}{a^2}-\frac{1}{b^2}\equiv\frac{b^2-a^2}{a^2b^2}.$$

Or, simplifying by Art. 182 before multiplying, we have

$$\frac{b+a}{ab}\times\frac{b-a}{ab}\equiv\frac{b^2-a^2}{a^2b^2}.$$

It is often better to multiply before simplifying.

Find the product of:

6. $\frac{3ax}{2by} \times \frac{2y}{3x^2}$.

7. $\frac{6ab}{cd} \times \frac{cx}{3b}$.

8. $\frac{5mn}{3am} \times \frac{2an}{10m}$.

9. $\frac{x-1}{a} \times \frac{x+1}{b}$.

10. $\frac{a}{a+b} \times \frac{b}{a-b}$.

11. $\frac{10b^2c^3}{9pq^2} \times \frac{3p^3q^2}{4b^2c}$.

12. $\frac{am}{5x} \times \frac{ma^n}{bm}$.

13. $\frac{x+1}{x-1} \times \frac{x-1}{x+1}$.

14. $\frac{x+1}{x-2} \times \frac{4x-8}{x^2-1}$.

15. $\frac{ax}{x+a} \times \left(\frac{x}{a} - \frac{a}{x}\right)$.

Simplify the following:

16. $\frac{x^3-y^3}{x^3+y^3} \times \frac{x+y}{x-y}$.

17. $\frac{a+b}{a-b} \times \frac{a^2-b^2}{a^3+b^3}$.

18. $\left(\frac{x+1}{y+1}\right)^3 \times \frac{y^3-1}{x^3+1}$.

19. $\left(1-\frac{1}{x}\right) \times \frac{1}{x^2-1}$.

20. $\left(1-\frac{y^4}{x^4}\right) \times \left(\frac{xy}{x^2+y^2}\right)$.

21. $\frac{a^3+b^3}{a^2-b^2} \times \frac{a-b}{a^2-ab+b^2}$.

22. $\left(x+\frac{1}{x-1}\right) \times \frac{2x-2}{x^3+1}$.

23. $\frac{x^2+5x+6}{x^2+6x+5} \times \frac{x^2+7x+10}{x^2+7x+12}$.

24. $\frac{x^2y}{a^2b} \times \frac{y^2z}{b^2c} \times \frac{abc}{xyz}$.

25. $\frac{x+y}{(x-y)^2} \times \frac{x-y}{(x+y)^2} \times \frac{x-y}{x+y}$.

26. $\frac{a^2}{5b^3} \times \frac{15b}{c} \times \frac{bc}{3a}$.

27. $\frac{m^2+mn}{m^2+n^2} \times \frac{m^3-n^3}{mn(m+n)}$.

28. $\frac{(a+b)(x+y)}{(a-b)(x-y)} \times \frac{a^2-b^2}{x+y}$.

29. $\left(1-\frac{2a}{1+a}\right)\left(1+\frac{2a}{1-a}\right)$.

30. $\frac{x^2-y^2}{x^2-3xy+2y^2} \times \frac{xy-2y^2}{x^2+xy} \times \frac{x^2-xy}{(x-y)^2}$.

31. $\frac{x+y}{x} \times \frac{x^2-y^2}{x^3+xy^2} \times \frac{4x^2}{(x+y)^2}$.

32. $\frac{x^2-14x-15}{x^2-4x-15} \times \frac{x^2-6x-27}{x^2-12x-45}$.

33. $\dfrac{a^2-x^2}{2ax} \times \dfrac{a^2x+ax^2}{a^2-2ax+x^2} \times \dfrac{2(a-x)}{a^2+ax}$.

34. $\dfrac{x(a-4)}{x^2+2ax+a^2} \times \dfrac{a(a+4)}{x^2-2ax+a^2} \times \dfrac{1}{ax}$.

35. $\dfrac{a+x}{(a-x)^2} \times \dfrac{a^2-x^2}{a^2+x^2} \times \dfrac{a^4-x^4}{(a+x)^3}$.

36. $\left(\dfrac{1-b^2}{1+c}\right) \times \left(\dfrac{1-c^2}{1+b}\right) \times \left(1+\dfrac{b}{1-b}\right)$.

37. $\dfrac{x^2-(a+b)x+ab}{x^2-(a+c)x+ac} \times \dfrac{x^2-c^2}{x^2-b^2}$.

38. $\left(\dfrac{a^2+1}{a+1}\right)^2 \times \left(\dfrac{a^2-1}{a+1}\right)^2 \times \left(\dfrac{a^2+1}{a-1}\right)^2$.

39. $\dfrac{a^3+b^3}{a^3-b^3} \times \dfrac{a^2-2ab+b^2}{a^2+2ab+b^2} \times \dfrac{a^2+ab+b^2}{a^2-ab+b^2}$.

40. $\dfrac{(x+y)^2-(z+w)^2}{(x+z)^2-(y+w)^2} \times \dfrac{(x-y)^2-(z-w)^2}{(x-z)^2-(w-y)^2}$.

DIVISION

187. The **Reciprocal** of a fraction is the fraction with its numerator and denominator interchanged.

Thus, $\frac{2}{3}$ is the reciprocal of $\frac{3}{2}$, and $\frac{b}{a}$ is the reciprocal of $\frac{a}{b}$.

188. The product of a fraction and its reciprocal is 1.

For, $$\frac{a}{b} \times \frac{b}{a} \equiv \frac{ab}{ba} = 1.$$

189. In algebra, as in arithmetic, the division of fractions is performed *by taking the product of the dividend and the reciprocal of the divisor.*

This is proved as follows:

Let $\frac{a}{b}$ and $\frac{c}{d}$ be any two fractions, and let x be the quotient obtained by dividing $\frac{a}{b}$ by $\frac{c}{d}$.

We are to show that $\frac{a}{b} \div \frac{c}{d} \equiv \frac{a}{b} \times \frac{d}{c}$.

Since $$x = \frac{a}{b} \div \frac{c}{d},$$

$$x \times \frac{c}{d} = \frac{a}{b}, \text{ by multiplying by } \frac{c}{d}.$$

$$\therefore x \times \frac{c}{d} \times \frac{d}{c} = \frac{a}{b} \times \frac{d}{c}, \text{ by multiplying by } \frac{d}{c}.$$

$$x = \frac{a}{b} \times \frac{d}{c}, \text{ since } \frac{c}{d} \times \frac{d}{c} = 1.$$

$$\therefore \frac{a}{b} \div \frac{c}{d} \equiv \frac{a}{b} \times \frac{d}{c}, \text{ by Axiom 1.}$$

Since $1 \div \frac{c}{d} \equiv 1 \times \frac{d}{c} \equiv \frac{d}{c}$, it is seen that the reciprocal of a fraction equals 1 divided by the fraction. Hence it follows that the divisor $\frac{c}{d}$ is contained $\frac{d}{c}$ times in 1, and $\frac{a}{b}$ *times* $\frac{d}{c}$, or $\frac{ad}{bc}$ times, in $\frac{a}{b}$.

Why is the divisor inverted? Why then multiply by the dividend? Integral divisors should be changed to the fractional form.

1. Divide $\frac{a^4 - 5a^3}{a - 6}$ by $\frac{a - 5}{a^3 - 6a^2}$.

$$\frac{a^4 - 5a^3}{a - 6} \div \frac{a - 5}{a^3 - 6a^2} \equiv \frac{a^3(a - 5)}{a - 6} \times \frac{a^2(a - 6)}{a - 5}$$

$$\equiv a^5, \text{ by canceling factors.}$$

When possible, always factor the expression and use cancellation.

2. Divide $\frac{x^3 - 1}{x^2 - a^2}$ by $\frac{x^2 + x + 1}{x - a}$.

$$\frac{x^3 - 1}{x^2 - a^2} \div \frac{x^2 + x + 1}{x - a} \equiv \frac{(x - 1)(x^2 + x + 1)}{(x + a)(x - a)} \times \frac{x - a}{x^2 + x + 1}$$

$$\equiv \frac{x - 1}{x + a}.$$

3. Divide $1 - \frac{y^2}{x^2}$ by $1 - \frac{2x}{y} + \frac{x^2}{y^2}$.

$$1 - \frac{y^2}{x^2} \equiv \frac{x^2 - y^2}{x^2}; \quad 1 - \frac{2x}{y} + \frac{x^2}{y^2} \equiv \frac{y^2 - 2xy + x^2}{y^2}.$$

$$\frac{x^2 - y^2}{x^2} \div \frac{y^2 - 2xy + x^2}{y^2} \equiv \frac{(x - y)(x + y)}{x^2} \times \frac{y^2}{(x - y)(x - y)}$$

$$\equiv \frac{y^2(x + y)}{x^2(x - y)}.$$

Perform the following divisions, simplifying the results:

4. $\dfrac{x+2}{x+1} \div \dfrac{x^2-4}{x^2-1}$.

5. $\dfrac{2a^2}{bc} \div \dfrac{3abx}{c^2y}$.

6. $\dfrac{a+b}{ab} \div \dfrac{2a}{a-b}$.

7. $\dfrac{a^3b^2}{x^5y^3} \div \dfrac{a^2b}{x^2y}$.

8. $\dfrac{a^2-121}{a^2-4} \div \dfrac{a+11}{a+2}$.

9. $\dfrac{x^4-y^4}{(x-y)^2} \div \dfrac{x^2+y^2}{x^2-y^2}$.

10. $\dfrac{2a^3b^2c^4}{3m^2x^3y^4} \div \dfrac{a^2b^2c^3}{m^3x^3y^4}$.

11. $\dfrac{x+y}{x^3-x^2y} \div \dfrac{xy+x^2}{xy-y^2}$.

12. $\dfrac{x^2-7x+12}{x^2+4x+4} \div \dfrac{x-4}{x+2}$.

13. $\dfrac{a^2-(b-c)^2}{(a^2-b^2)^2} \div \dfrac{a-b+c}{a^4-b^4}$.

14. $\dfrac{(x+y)^2-z^2}{(x-y)^2-z^2} \div \dfrac{x+y+z}{x-y-z}$.

15. $\dfrac{a^3-b^3}{a^2+ab+b^2} \div \dfrac{a-b}{a+b}$.

16. $\dfrac{ab-b^2}{(a+b)^2} \div \dfrac{b}{a^2-b^2}$.

17. $\dfrac{a^3-x^3}{a^3+x^3} \div \dfrac{(a-x)^2}{a^2-x^2}$.

18. $\dfrac{x^3-1}{x^2+x+1} \div \dfrac{x-1}{x+1}$.

19. $\left(\dfrac{1}{b^2}-\dfrac{1}{a^2}\right) \div \left(\dfrac{b}{a}+1\right)$.

20. $\left(\dfrac{x}{y}+1\right) \div \left(\dfrac{x^2}{y}+\dfrac{y^2}{x}\right)$.

21. $\dfrac{x^2-x}{x^3+1} \div \dfrac{x^2-2x+1}{x^3-x^2+x}$.

22. $\left(\dfrac{x^2}{x^2-y^2}-1\right) \div \left(\dfrac{y^2}{x^2-y^2}+1\right)$.

23. $\dfrac{a^{3m}-1}{x^{3a}+1} \div \dfrac{a^{2m}-1}{x^{2a}-1}$.

24. $\dfrac{1+x-x^2-x^4}{1-y^2} \div \dfrac{x^2-1}{y^2-1}$.

25. $\left(y-x+\dfrac{x^2}{y}\right) \div \left(\dfrac{x}{y^2}+\dfrac{y}{x^2}\right)$.

26. $\left(\dfrac{x^2}{a^2}+1+\dfrac{a^2}{x^2}\right) \div \left(\dfrac{x}{a}-1+\dfrac{a}{x}\right)$.

27. $\dfrac{x^2-4x-5}{x^2-3x+2} \div \dfrac{x^2-3x-10}{2x^2-7x+3}$.

28. $\dfrac{x^2-(y-z)^2}{y^2-(x+z)^2} \div \dfrac{ay-ax-az}{bx+by+bz}$.

29. $\left(\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{a^2-b^2}{a^2+b^2}\right) \div \left(\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}\right)$.

30. $\dfrac{(a+b)^2-(c+d)^2}{(a+c)^2-(b+d)^2} \div \dfrac{(a-c)^2-(d-b)^2}{(a-b)^2-(d-c)^2}$.

COMPLEX FRACTIONS

190. A fractional expression having a fractional numerator or denominator, or both, is called a **Complex Fraction.**

Thus, $\dfrac{\frac{a}{b}}{\frac{c}{d}}$ and $\dfrac{1-\frac{a}{x}}{ax}$ are complex fractions. The former is equivalent to $\frac{a}{b} \div \frac{c}{d}$, the latter to $\left(1-\frac{a}{x}\right) \div ax$.

Simplify the complex fraction $\dfrac{\frac{x^2-y^2}{x}}{\frac{x+y}{y}}$.

$$\frac{x^2-y^2}{x} \div \frac{x+y}{y} \equiv \frac{(x+y)(x-y)}{x} \times \frac{y}{x+y}$$

$$\equiv \frac{y(x-y)}{x}.$$

191. Instead of performing the division indicated, it is sometimes better to multiply both numerator and denominator by a factor that will render them integral. The L. C. M. of their denominators is such a factor.

Thus, in the preceding example, the L. C. M. of the denominators is xy. Multiplying by this, we have

$$\frac{y(x^2-y^2)}{x(x+y)} \equiv \frac{y(x-y)}{x}, \text{ by canceling } x+y.$$

1. Simplify $\dfrac{a-\frac{ax}{a+x}}{a+\frac{ax}{a-x}}$.

$$\frac{a-\frac{ax}{a+x}}{a+\frac{ax}{a-x}} \equiv \frac{\frac{a^2+ax-ax}{a+x}}{\frac{a^2-ax+ax}{a-x}}$$

$$\equiv \frac{\frac{a^2}{a+x}}{\frac{a^2}{a-x}} \equiv \frac{a^2(a-x)}{a^2(a+x)} \equiv \frac{a-x}{a+x}.$$

Simplify the following complex fractions:

2. $\dfrac{x+\dfrac{a^2}{x}}{x-\dfrac{a^4}{x^3}}$.

3. $\dfrac{1+\dfrac{b}{a-b}}{1-\dfrac{b}{a+b}}$.

4. $\dfrac{a+\dfrac{ab}{a-b}}{a-\dfrac{ab}{a+b}}$.

5. $\dfrac{\dfrac{1}{m}-\dfrac{1}{n}}{\dfrac{n}{m}-\dfrac{m}{n}}$.

6. $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{b}{a}+\dfrac{a}{b}}$.

7. $\dfrac{\dfrac{a}{b}-\dfrac{c}{d}}{\dfrac{a}{b}+\dfrac{c}{d}}$.

8. $\dfrac{3x-2+\dfrac{1}{x}}{\dfrac{3x-1}{x}}$.

9. $\dfrac{1+\dfrac{x}{1+x}}{x+\dfrac{1}{1+x}}$.

10. $\dfrac{\dfrac{1}{1-a}+\dfrac{1}{1+x}}{\dfrac{1}{1-a}-\dfrac{1}{1+x}}$.

11. $\dfrac{\dfrac{x}{x-y}-\dfrac{x}{x+y}}{\dfrac{y}{x-y}-\dfrac{x}{x+y}}$.

12. $\dfrac{\dfrac{x}{1+x}+\dfrac{1-x}{x}}{\dfrac{x}{1+x}-\dfrac{1-x}{x}}$.

13. $\dfrac{x-1+\dfrac{6}{x-6}}{x-2+\dfrac{3}{x-6}}$.

14. $\dfrac{\dfrac{a}{a+x}-\dfrac{a}{a-x}}{\dfrac{a}{a-x}+\dfrac{a}{a+x}}$.

15. $\dfrac{\dfrac{x}{x-y}-\dfrac{x}{x+y}}{\dfrac{y}{x-y}+\dfrac{x}{x+y}}$.

16. $\dfrac{x-z+\dfrac{1}{x+z}}{x+z+\dfrac{1}{x-z}}$.

17. $\dfrac{\dfrac{a}{b}-\dfrac{a-b}{a+b}}{\dfrac{a}{b}+\dfrac{a+b}{a-b}}$.

18. $\dfrac{\dfrac{a-b}{a+b}+\dfrac{a^2+b^2}{a^2-b^2}}{\dfrac{a^2}{a-b}+\dfrac{a^2b+b^3}{(a-b)^2}}$.

192. Continued Fractions.

Expressions of the form $\cfrac{a}{b+\cfrac{c}{d+\cfrac{e}{f+\cdots}}}$ are called **Continued Fractions.**

193. Continued fractions are usually simplified by first reducing the *last complex part* to a simple fraction; then the last complex part of the resulting fraction to a simple fraction, and so on. This is best done by multiplying the numerator and

denominator of the last complex part by the last denominator, and so on from below upward.

1. Simplify $\dfrac{1}{a+\dfrac{1}{a+\dfrac{1}{a}}}$.

$$\frac{1}{a+\dfrac{1}{a+\dfrac{1}{a}}} \equiv \frac{1}{a+\dfrac{a}{a^2+1}},$$ by multiplying numerator and denominator of last complex part by a.

$$\equiv \frac{a^2+1}{a(a^2+1)+a},$$ by multiplying resulting numerator and denominator by a^2+1.

$$\equiv \frac{a^2+1}{a(a^2+2)}.$$

Simplify the following:

2. $\dfrac{x}{x-\dfrac{x+2}{x+2-\dfrac{x+1}{x}}}$.

3. $\dfrac{1}{x-\dfrac{x^2-1}{x+\dfrac{1}{x-1}}}$.

4. $\dfrac{x^2-4}{x+2-\dfrac{x}{x-\dfrac{x-1}{x+2}}}$.

5. $\dfrac{a+b}{a+b+\dfrac{1}{a-b+\dfrac{1}{a+b}}}$.

MISCELLANEOUS EXERCISES

194. 1. Reduce $\dfrac{2x^3-4x^2-13x-7}{6x^3-11x^2-37x-20}$ to its lowest denominator.

2. Reduce $\dfrac{2x^3+7x^2-9x-9}{2x^3+9x^2+x-3}$ to its lowest denominator.

Simplify:

3. $\dfrac{x+y}{2x-2y}+\dfrac{x-y}{2x+2y}-\dfrac{4x^2y^2}{x^4-y^4}$.

4. $\dfrac{1}{1+\dfrac{x}{y+z}}+\dfrac{1}{1+\dfrac{y}{x+z}}+\dfrac{1}{1+\dfrac{z}{x+y}}$.

5. $\left(\frac{a^2}{x}-\frac{x^2}{a}\right)\left(\frac{a}{x^2}+\frac{x}{a^2}\right)\div\left(\frac{a^2}{x}+\frac{x^2}{a}\right)\left(\frac{a}{x^2}-\frac{x}{a^2}\right)$.

6. $\left(\frac{x^2}{y^2}+\frac{x}{y}+1\right)\times\left(\frac{x^2}{y^2}-\frac{x}{y}+1\right)$.

7. $\dfrac{\frac{a}{c^2}+\frac{c}{a^2}}{\frac{1}{a^2}-\frac{1}{ac}+\frac{1}{c^2}}$.

8. $\dfrac{\frac{x^3}{y}+y^2}{x^2-\frac{y^3}{x}}\div\dfrac{xy+y^2}{x(x-y)}$.

9. $\dfrac{\frac{y+1}{y-1}+\frac{y-1}{y+1}}{\frac{y+1}{y-1}-\frac{y-1}{y+1}}$.

10. $\dfrac{1}{x^2-\dfrac{x^3-1}{x+\frac{1}{x+1}}}$.

11. $\dfrac{1+\frac{a-b}{a+b}}{1-\frac{a-b}{a+b}}\div\dfrac{1+\frac{a^2-b^2}{a^2+b^2}}{1-\frac{a^2-b^2}{a^2+b^2}}$.

12. $m^2-2n^2-\dfrac{6n^3-m^2n+mn^2}{m-3n}$.

13. $\dfrac{x+y}{x-y}-\dfrac{x-y}{x+y}-\dfrac{4y^2}{x^2-y^2}$.

14. $\dfrac{5a-4}{9}+\dfrac{12a+2}{11a-8}-\dfrac{10a+17}{18}$.

15. $\dfrac{1}{4x^3(x+a)}+\dfrac{1}{4x^3(x-a)}+\dfrac{1}{2x^2(x^2+a^2)}$.

16. $\dfrac{(a+b+c)^2+(a-b)^2+(b-c)^2+(c-a)^2}{a^2+b^2+c^2}$.

17. $\left(1-\dfrac{2ab}{a^2+b^2}\right)\div\left(\dfrac{a^3-b^3}{a-b}-3ab\right)$.

18. $\left(p^2-q^2-r^2-2qr\right)\div\dfrac{p+q+r}{p+q-r}$.

Expand the following:

19. $\left(\frac{x}{y}+1+\frac{y}{x}\right)\left(\frac{x}{y}-1+\frac{y}{x}\right)$.

20. $\left(\frac{a(b^3-a^3)}{b(b+a)}\right)\left(\frac{(b^2-a^2)^2}{b^2+ab+a^2}\right)\left(\frac{(b+a)^2}{(b-a)^2}\right)$.

21. $\left(\frac{x^4+x^2y^2+y^4}{x^6-y^6}\right)\left(\frac{x+y}{x^3+y^3}\right)\left(\frac{x^2-y^2}{x}\right)$.

Simplify the following:

22. $\left(\frac{a+b}{a-b}+\frac{a-b}{a+b}\right)\div\left(\frac{a+b}{2(a-b)}-\frac{a-b}{2a+2b}\right)$.

23. $\left(\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}\right)\div\left(\frac{x-y}{x+y}-\frac{x^2+y^2}{x^2-y^2}\right)$.

24. $\frac{1}{x+y}+\frac{1}{y+z}+\frac{1}{z+x}-\frac{(x+y+z)^2}{(x+y)(y+z)(z+x)}$.

25. $\dfrac{\frac{x^3-y^3}{x^3+y^3}\left(1-\frac{2x}{x+y}\right)}{1+\frac{2xy}{x^2-xy+y^2}}$.

26. $\dfrac{a}{1-\dfrac{a}{1+a+\dfrac{a}{1-a+a^2}}}$.

27. $\frac{a^2-(b-c)^2}{(a+c)^2-b^2}+\frac{b^2-(c-a)^2}{(b+a)^2-c^2}+\frac{c^2-(a-b)^2}{(c+b)^2-a^2}$.

28. $\left(\frac{x+y}{x-y}+\frac{x^2+y^2}{x^2-y^2}\right)\div\left(\frac{a-b}{a+b}-\frac{a^3-b^3}{a^3+b^3}\right)$.

29. $\frac{x+y}{(y-z)(z-x)}+\frac{y+z}{(z-x)(x-y)}+\frac{z+x}{(x-y)(y-z)}$.

30. $\frac{a(b+c)}{(a-b)(c-a)}+\frac{b(c+a)}{(b-c)(a-b)}+\frac{c(a+b)}{(c-a)(b-c)}$.

31. $\frac{xy}{(z-x)(z-y)}+\frac{yz}{(x-y)(x-z)}+\frac{zx}{(y-z)(y-x)}$.

Find the value of:

32. $\frac{c}{2mc-2my}+\frac{d}{2md-2my}$ when $y=\frac{1}{2}(c+d)$.

SIMPLE EQUATIONS

WITH ONE UNKNOWN NUMBER

195. As already seen, an **Equation** is an equality that exists only for particular values of the letters representing the *unknown numbers.*

Thus, $x - 2 = 0$ is true only for the value $x = 2$.

196. An equation is said to be **Rational** when, after uniting like terms, the two members are rational with respect to the unknown numbers. In the contrary case it is said to be **Irrational.**

Thus, $x + 3 = 0$ is rational, but $\sqrt{x} + 3 = 8$ is irrational.

197. An equation is said to be **Integral** when, after uniting like terms, the two members are integral with respect to the unknown numbers. In the contrary case it is said to be **Fractional.**

Thus, $3x + 5 = 0$ is integral, but $\frac{1}{x} + 2 = 3$ is fractional.

198. A rational integral equation in which no unknown number is of higher degree than the first is called a **Simple** or **Linear Equation.** It is also called an equation of the *first degree.*

Thus, $5x + 3 = 18$ is a simple or linear equation.

199. In solving equations we have two kinds of numbers to consider — numbers whose values are either given or supposed to be given, and numbers whose values are to be found. The former are called **Known Numbers**; the latter, **Unknown Numbers.**

200. The *unknown numbers* in an equation are usually represented by the *last letters* of the alphabet, particularly by x, y, z.

Numbers whose values are *known* are usually represented by figures; if only *supposed* to be known, by the first letters of the alphabet, as a, b, c.

Thus, in the equation $x + a = 0$, a is supposed to be known, and $x = -a$ satisfies the equation, just as the equation $x + 2 = 0$ is satisfied if $x = -2$. What is meant by an equation being *satisfied?* What is meant by the *root* of the equation $x + a = 0$?

201. Equivalent Equations.

Two equations are said to be **Equivalent** when each has all the roots of the other.

Thus, $2x - 1 = 3$ and $4x = 8$ are equivalent equations, for $x = 2$ is a root, and the only root of each. But $x = 2$ and $x^2 = 4$ are not equivalent, for $x = -2$ is one root of the second, but it is not a root of the first.

202. A single equation is often equivalent to two or more equations.

Thus, the equation $x^2 + x - 6 = 0$

may be written $(x + 3)(x - 2) = 0$;

and this equation is equivalent to the two equations

$$x + 3 = 0 \text{ and } x - 2 = 0;$$

for the root of the first equation is $x = -3$ and that of the second, $x = 2$, and both these roots satisfy the original equation.

In solving $x^2 + x - 6 = 0$, we write it as $x + 3 = 0$ and $x - 2 = 0$, and solve each of these equations. Hence, it is seen that equations of higher degree than the first can often be solved by finding the equivalent linear equations, as in the case of equations solved by factoring (Art. 161).

203. In the solution of equations, each step follows from the preceding one by application of the axioms already considered; that is, the given equation is first converted into a similar equivalent one, then this equation into a still simpler one, and so on; in the successive steps, each of the equivalent equations has the *same* root or roots as the original one.

To illustrate, consider the solution of the equation $2x - 3 = 5$:

$$2x - 3 = 5$$

is equivalent to $2x = 8$, by adding 3 to both members;

that is, to $x = 4$, by dividing by 2.

Here it is evident that 4 is a root of each of the equivalent equations $2x - 3 = 5$, $2x = 8$, and $x = 4$.

204. It will be seen that only such operations are allowed to be performed upon both members of an equation as will convert it into an *equivalent* equation, that is, into an equation that has the *same* root or roots as the original one. For this reason it will tend to clearness in subsequent work, as well as to safeguard the student against wrong results if we give attention at this stage to certain necessary limitations on the use of the axioms as applied in the solution of equations.

1. By Axiom 6 (Art. 36), if the two members of an equation are multiplied by equal numbers, the results are *equal.* This is true as to the equality of the members — the resulting equation being true when the original is true; but it does not necessarily follow that the resulting equation is *equivalent* to the given one.

For example, if the two members of the equation $x + 2 = 5$, whose only root is 3, are multiplied by $x - 2$, we have

$$(x + 2)(x - 2) = 5(x - 2),$$

or

$$x^2 - 5x + 6 = 0,$$

which may be written $(x - 3)(x - 2) = 0$.

Obviously this equation has two roots, $x = 3$ and $x = 2$, while the original equation has only the root $x = 3$; that is, in multiplying both members by $x - 2$, we have introduced a new root 2, and this is the root obtained by placing the factor $x - 2$ equal to 0.

Hence, it appears that in general, *multiplying both members of an integral equation by a factor containing an unknown number of the equation introduces one or more new roots,* "*extraneous roots,*" as they are called.

However, multiplying both members of a *fractional* equation by such a factor does not, in general, introduce new roots when the multiplier occurs in the equation as a denominator of a fraction having its lowest denominator.

Thus, $\frac{2}{x-2} = x - 3$, when multiplied by $x - 2$, becomes $2 = (x - 3)(x - 2)$, or $x^2 - 5x + 4 = 0$, which may be written $(x - 1)(x - 4) = 0$, and this has two roots, $x = 1$ and $x = 4$, both of which satisfy the original equation. The root $x = 2$, which is the only root that could be introduced by the multiplier $x - 2$, does not appear.

2. Again, by the axiom referred to above, we should be permitted to multiply the two members of the equation by 0. But this is inadmissible, for multiplying both members of the equation $x - 2 = 0$ by 0, we have

$$0(x - 2) = 0.$$

But *any value* of x will be a root of this equation, since

$$\begin{aligned} &\text{when } x = 1,\ 0(1 - 2) = 0; \\ &\text{when } x = 2,\ 0(2 - 3) = 0; \\ &\text{when } x = 3,\ 0(3 - 2) = 0;\ \text{etc.} \end{aligned}$$

3. By Axiom 7, if the two members of an equation are divided by equal numbers (not zero), the results are *equal*. This is also true, but it does not necessarily follow that the resulting equation is *equivalent* to the given one.

For example, if the two members of the equation

$$(x + 2)(x - 2) = 5(x - 2)$$

are divided by $x - 2$, we have $x + 2 = 5$, whose only root is 3, whereas the original equation has this root, and also the root $x = 2$, as seen in 1 above; that is, in dividing both members by $x - 2$, we have lost the root 2, and this is the root obtained by placing the divisor $x - 2$ equal to 0.

Hence, it appears that in general, *dividing both members of an equation by an expression containing an unknown number of the equation loses one or more of its roots.*

205. From these considerations we are led to state the legitimate operations which may be performed on the two members of an equation, as follows:

I. **If the same number or expression is added to or subtracted from both members of an equation, the result is an equivalent equation.**

II. **If both members of an equation are multiplied or divided by the same factor, which is not zero, and which does not contain an unknown number of the equation, the result is an equivalent equation.**

Proofs of these propositions are given later (Art. 211).

Changing the signs of all the terms of an equation does not destroy the equality, for this is equivalent to multiplying both members by -1 (Prin. II).

206. A simple equation of one unknown number has *one root and only one.*

For by Principles I and II, any simple equation can be reduced first to the form (1) $ax = b$, and then to (2) $x = \frac{b}{a}$. But $\frac{b}{a}$ gives but *one result*, and hence but one root; and since (1) is equivalent to (2), and each is equivalent to the original equation, the original equation has but one root.

INTEGRAL EQUATIONS

207. The several steps in the practical solution of the simple equation involving one unknown number have already been set forth (Arts. 38–40). These results may be summarized in the following general directions:

1. *Transpose the terms containing the unknown number to the first member, and all the other terms to the second;*

2. *Unite like terms;*

3. *Divide both members by the coefficient of the unknown number.*

It is important that the beginner in all cases check his solution by showing that it renders the given equation an *identity.*

208. Hitherto we have solved only equations in which the known numbers were represented by figures. The solution of equations in which the numbers supposed to be *known* are represented by letters involves no new principle, and will be understood from a comparison of the following illustrative solutions:

1. Given the equation $3x - 8 = x + 2$, to find the value of x.

$$3x - 8 = x + 2.$$

Transposing, $3x - x = 2 + 8.$ (Axs. 2, 3)

Uniting terms, $2x = 10.$

Dividing by 2, $x = 5.$ (Ax. 7)

CHECK. Substituting 5 for x in the original equation, we have $15 - 8 = 5 + 2$, an *identity*.

2. Given the equation $3ax - 4a^2 = ax + 2a^2$, to find the value of x.

$$3ax - 4a^2 = ax + 2a^2.$$

Transposing, $3ax - ax = 2a^2 + 4a^2.$

Uniting terms, $2ax = 6a^2.$

Dividing by $2a$, $x = 3a.$

CHECK. Substituting $3a$ for x in the original equation, we have $9a^2 - 4a^2 \equiv 3a^2 + 2a^2$.

Solve the following equations:

3. $3x + 5 = x + 17.$
4. $14x + 6x = 72 + 8x.$
5. $12 - 3x = 5x - 4.$
6. $x + 23 = 5 + 7x.$
7. $11x - 16 = 3x - 31.$
8. $x - 20 = 85 - (x - 25).$
9. $3(x + 8) = 14(77 - x).$
10. $1 - 2[x - 3(1 + x)] = 0.$
11. $5(x - 2) - 6(x + 4) = 21.$
12. $ax - b = 0.$
13. $bx - b^2 = cx - c^2.$
14. $ax - b(x - 1) = c.$
15. $(a - b)x = c + d.$
16. $bx + ax - ac = m.$
17. $3(x + 1) = -5(x - 1).$
18. $5a = 3(2x - a) - 4a.$
19. $(a - b)x = a^2 - (a + b)x.$
20. $ax - bx = (a + b)(a - b).$
21. $x - 2(x - 3) + 1 = x - 5.$
22. $3(x - 1)(x + 1) = x(3x + 4).$

23. $12x + 2(x+2) = 60 + 3(x-4)$.

24. $2(x-b) + 3(x-2b) = 2b$.

25. $(x+1)(x+2) = (x-2)(x+4)$.

26. $6x - [7x - (8x - 18)] = 16$.

27. $10(2x-9) - 7(4x-19) + 5 = 4x - 3(2x-3)$.

28. $4x^3 - 4(x^3 - x^2 + x - 2) = 4x^2$.

29. $x - 3a^2 - 192a^2b^3 = 4bx + 16b^2x$.

30. $2(x-2) + 3(x-3) + 4(x-4) = 3x + 7$.

31. $3x^2 = (x+1)^2 + (x+2)^2 + (x+3)^2$.

32. $(a^2 + x)(b^2 + x) = (ab + x)^2$.

33. $(2x+1)(x-3) + 7 = x - 2(x-4)(2-x)$.

34. $3x - [3x - (3 + 2x)] = 2(3 - \frac{1}{2}x)$.

35. $(x-a)^3 + (x-b)^3 + (x-c)^3 = 3(x-a)(x-b)(x-c)$.

EQUATIONS CONTAINING FRACTIONS

209. If the coefficients of a simple equation are fractional, or what is the same thing, if *known* numbers occur in the denominators of any of its terms, the equation can be reduced to an equivalent one in which the coefficients are all integral by multiplying both members by the L. C. M. of the denominators. Many equations that are *fractional as they stand* are by the removal of denominators reduced to equivalent ***simple*** equations and solved as such, as is seen in several of the examples which follow.

210. The process of reducing an equation containing fractions to an equivalent equation without fractions is called **Clearing of Fractions.**

Fractions should be reduced to *lowest* denominators before multiplying by the L. C. M. of the denominators; otherwise one or more extraneous roots may be introduced (Art. 204, 1).

In some cases the fractions can be united to advantage before clearing.

1. Given the equation $\frac{x-1}{3}-\frac{2x-3}{5}=\frac{3x-4}{2}-x$, to find the value of x.

$$\frac{x-1}{3}-\frac{2x-3}{5}=\frac{3x-4}{2}-x.$$

Multiplying both members by 30,

$$10(x-1)-6(2x-3)=15(3x-4)-30x, \qquad \text{(Ax. 6)}$$

or, removing parentheses,

$$10x-10-12x+18=45x-60-30x.$$

Transposing and uniting terms,

$$-17x=-68.$$

Dividing by -17, $x=4.$ (Ax. 7)

CHECK. $\frac{3}{5}-\frac{5}{5}=\frac{8}{2}-4$, or $1-1=4-4$.

2. Solve the equation $\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}$.

$$\frac{4x+3}{9}+\frac{7x-29}{5x-12}=\frac{8x+19}{18}.$$

Multiplying both members by 18,

$$8x+6+\frac{18(7x-29)}{5x-12}=8x+19.$$

Transposing and uniting terms,

$$\frac{18(7x-29)}{5x-12}=13.$$

Multiplying by $5x-12$,

$$18(7x-29)=13(5x-12),$$

or

$$126x-522=65x-156.$$

Again transposing and uniting terms,

$$61x=366.$$

Dividing by 61, $x=6.$

CHECK. $\frac{27}{9}+\frac{13}{18}=\frac{67}{18}$.

In this case it is better to unite the fractions having numerical denominators before clearing.

Is any extraneous root introduced by multiplying by $5x-12$? Why not?

3. Solve the equation $\frac{x+3}{x+5}-\frac{x+1}{x+3}=\frac{x-5}{x-7}-\frac{x-3}{x-5}$.

The denominators are such as to suggest adding the fractions in each member separately before clearing. Then

$$\frac{x^2+6x+9-x^2-6x-5}{(x+5)(x+3)}=\frac{x^2-10x+25-x^2+10x-21}{(x-7)(x-5)}.$$

$$\therefore \frac{4}{(x+5)(x+3)}=\frac{4}{(x-7)(x-5)}.$$

Dividing by 4 and clearing of fractions,

$$x^2-12x+35=x^2+8x+15.$$

$$\therefore -20x=-20.$$

$$\therefore x=1.$$

Solve this example by reducing the fractions to mixed expressions before clearing. This method is often preferable when the degree of the numerator equals or *exceeds* that of the denominator.

4. Solve the equation $\frac{x^2-2x+1}{x^2-1}=\frac{x-6}{x-3}$.

Writing the first member in the factored form, we have

$$\frac{(x-1)^2}{(x+1)(x-1)}=\frac{x-6}{x-3}.$$

Canceling $x-1$, $$\frac{x-1}{x+1}=\frac{x-6}{x-3}.$$

Multiplying by $(x+1)(x-3)$,

$$(x-1)(x-3)=(x-6)(x+1),$$

or $$x^2-4x+3=x^2-5x-6.$$

$$\therefore x=-9.$$

In this case it is not legitimate to multiply by $(x^2-1)(x-3)$, for then

$$(x^2-2x+1)(x-3)=(x-6)(x^2-1),$$

or $$x^3-5x^2+7x-3=x^3-6x^2-x+6,$$

or $$x^2+8x-9=0, \text{ whence } (x-1)(x+9)=0.$$

Solving, $$x=1, \text{ or } -9.$$

But $x=1$ does not satisfy the original equation. It was introduced by multiplying by the unnecessary factor $x-1$ of the L. C. D. contained in the numerator x^2-2x+1. Fractions should first be reduced to lowest denominators before clearing.

It is always advisable to see if the fractions can be simplified before clearing.

Solve the equations:

5. $x+\frac{x}{4}=15.$

6. $\frac{3}{x+2}=\frac{2}{x+1}.$

7. $2x+\frac{x}{3}=14.$

8. $\frac{x-2}{2x-5}=\frac{2}{3}.$

9. $\frac{7+x}{23+x}=\frac{59}{67}.$

10. $\frac{2x-1}{2(x-3)}=\frac{3(x-2)}{3x-1}.$

11. $\frac{z}{3}+\frac{z}{4}+\frac{z}{6}=18.$

12. $\frac{\frac{2}{3}(4x-1)}{\frac{3}{4}(5x+1)}=\frac{2}{3}.$

13. $\frac{7x}{x-4}-\frac{x}{x-5}=6.$

14. $x-\frac{x}{4}-\frac{x}{3}=5.$

15. $\frac{m}{x-a}=\frac{n}{x-b}.$

16. $2y+\frac{y}{4}=2+\frac{5y}{4}.$

17. $\frac{m+x}{n+x}=\frac{m+1}{n+1}.$

18. $\frac{x+m}{x-m}=\frac{x+n}{x-n}.$

19. $\frac{x+2}{x-2}-\frac{x-4}{x+4}=\frac{12}{x}.$

20. $\frac{2}{x}-\frac{3}{x}=\frac{a}{x}-c.$

21. $\frac{5}{x^2-1}-\frac{2}{x+1}=\frac{1}{x-1}.$

22. $\frac{2x}{a}+\frac{2x+1}{a+1}=3.$

23. $\frac{5-x}{12}-\frac{x-4}{15}=\frac{x-3}{20}.$

24. $\frac{y+3}{4}-\frac{y-8}{5}=\frac{y-5}{2}-1.$

25. $\frac{8}{x+3}-\frac{15}{7x+2}=\frac{9}{2x+6}.$

26. $\frac{6x+7}{15}-\frac{2(x-1)}{7x-6}=\frac{2x+1}{5}.$

27. $\frac{x^2-x+1}{x-1}+\frac{x^2+x+1}{x+1}=2x.$

28. $x+1-\frac{x^2-x-6}{x+2}=\frac{x+1}{x-5}.$

29. $\frac{x-1}{x-2}+\frac{x+1}{x+2}=\frac{2(x^2+4x+1)}{(x+2)^2}.$

30. $\frac{x-1}{x-2}-\frac{x-2}{x-3}=\frac{x-3}{x-4}-\frac{x-4}{x-5}.$

31. $\frac{1}{x+7}-\frac{1}{x+1}=\frac{1}{x+1}-\frac{1}{x+3}.$

32. $\frac{x-12}{x-7}+\frac{x-4}{x-12}=2+\frac{7}{x-7}.$

33. $\dfrac{m-n}{x-a}+\dfrac{n-p}{x-b}+\dfrac{p-m}{x-c}=0.$

34. $\dfrac{x-c}{x-b}=\dfrac{x-b}{x-c}+\dfrac{2(b-c)}{x-b-c}.$

35. $\dfrac{mx}{m+x}+\dfrac{nx}{n+x}=m+n.$

36. $\dfrac{a}{x-a}-\dfrac{b}{x+a}=\dfrac{b}{x^2-a^2}.$

37. $\dfrac{1}{m(n-x)}+\dfrac{1}{n(p-x)}=\dfrac{1}{m(p-x)}.$

38. $\dfrac{x-5}{x+5}-\dfrac{x-10}{x+10}=\dfrac{x-4}{x+4}-\dfrac{x-9}{x+9}.$

39. $\dfrac{x+1}{x+2}+\dfrac{x+6}{x+7}=\dfrac{x+2}{x+3}+\dfrac{x+5}{x+6}.$

40. $\dfrac{2x-3}{x-4}+\dfrac{3x-2}{x-8}=\dfrac{5x^2-29x-4}{x^2-12x+32}.$

41. $\dfrac{2x+1}{2x-16}-\dfrac{2x-1}{2x+12}=\dfrac{9x+17}{x^2-2x-48}.$

42. $\dfrac{x-1}{x^3-8}=\dfrac{3}{x-2}+\dfrac{1-3x}{x^2+2x+4}.$

43. $\dfrac{3}{3x-2}-\dfrac{2}{2x-3}=\dfrac{2}{2x+3}-\dfrac{3}{3x+2}.$

44. $\dfrac{x-b}{c+a}+\dfrac{x-a}{c+b}+\dfrac{x-c}{c+b}=3.$

45. $\dfrac{z^2-az-bz+ab}{z-a}=\dfrac{z^2-2bz+2b^2}{z-b}-\dfrac{c^2}{z-c}.$

46. $\dfrac{x-a}{b}-\dfrac{x^2-bx-a^2}{bx-a^2}=1+\dfrac{a^2}{bx-a^2}.$

47. $\dfrac{\dfrac{mx-n}{x}}{\dfrac{m}{n}}=\dfrac{\dfrac{mx+n}{x}}{\dfrac{n}{m}}.$

48. $\dfrac{1}{2}+\dfrac{\frac{1}{2}}{1+\dfrac{1}{y}}=\dfrac{1+\dfrac{5}{y}}{1+\dfrac{6}{y}}.$

49. $\dfrac{y+a}{y-a}-\dfrac{y-b}{y+b}=\dfrac{2(a+b)}{y}.$

50. $\dfrac{z-b}{z+2b}+\dfrac{z+7b}{z+b}=\dfrac{z-b}{z-3b}+\dfrac{z+7b}{z+6b}.$

PROOFS OF IMPORTANT THEOREMS

211. The following laws governing the solutions of equations have already been stated and illustrated by examples. The proofs of these laws will now be given.

These may be omitted by beginners if desired.

I. THEOREM. **If the same number or expression is added to or subtracted from both members of an equation, the result is an equivalent equation.**

Let $$A = B \tag{1}$$

be any equation, and C any number or expression.

It is to be proved that

$$A + C = B + C \tag{2}$$

is equivalent to the given equation.

Since $C \equiv C$, it is evident that any values of the unknown number that make $A = B$ an identity also make $A + C = B + C$ an identity. But the roots of (1) are such values.

Hence, all the roots of (1) are the roots of (2).

For the same reason,

$$A + C - C = B + C - C \tag{3}$$

has all the roots of (2). But since $C - C = 0$, (3) may be written $A + 0 = B + 0$, that is, $A = B$.

Hence, all the roots of (2) are roots of (1).

And since (2) has all the roots of (1), and (1) has all the roots of (2), it follows that the two equations are equivalent.

Show that $A - C = B - C$ is equivalent to the given equation.

II. THEOREM. **If both members of an equation are multiplied or divided by the same factor, which is not zero, and does not contain an unknown number of the equation, the result is an equivalent equation.**

Let $$A = B \tag{1}$$

be any equation, and C any factor, which is not zero, and does not contain the unknown number or numbers.

It is to be proved that

$$AC = BC \tag{2}$$

is equivalent to the given equation.

By Theorem I, (1) and (2) are respectively equivalent to the equations,

$$A - B = 0, \tag{3}$$

$$C(A - B) = 0. \tag{4}$$

It is sufficient therefore to prove that (3) and (4) are equivalent.

Equation (4) is evidently satisfied by any value of the unknown number that will make either factor equal to zero, that is, by any values that make $A - B = 0$, or $C = 0$, or both, and by no other values. But the values that make $A - B = 0$, are the roots of (1), the original equation.

Hence, all the roots of (1) are the roots of (2).

Conversely, every root of (2), making $C(A - B) = 0$, must also make $A - B = 0$, because C is not zero, and cannot become zero, since it does not contain the unknown number.

Hence, all the roots of (2) are the roots of (1).

And since (2) has all the roots of (1), and (1) has all the roots of (2), it follows that the two equations are equivalent.

The case for the division of an equation by the same number is similarly proved.

(*a*) If $C = 0$, then *any value* of the unknown number would be a root of the resulting equation, but not of the given one (Art. 204, 2). For this reason the value $C = 0$ must be excluded.

(*b*) If C contains the unknown number, then (2) has the roots of $C = 0$ in addition to the roots of $A - B = 0$. But the roots of $C = 0$ do not satisfy (1). Hence C must be independent of the unknown number.

(*c*) The present discussion is limited to *finite* values, but it will appear hereafter that the multiplier must not become *infinitely great*, that is, C must not equal "infinity" (Art. 452).

III. THEOREM. **If both members of a rational fractional equation, after the fractions are reduced to lowest denominators, are multiplied by the lowest common denominator of the fractions, the result is, in general, an equivalent equation.**

By Theorem I the equation can be reduced to the equivalent form in which the second member is 0.

Then by uniting the fractions of the first member and reducing the resulting fraction to its lowest denominator, the given equation may be transformed into an equivalent one of the form

$$\frac{A}{B} = 0, \tag{1}$$

in which A is prime to B, and B is the lowest common denominator of the fractions.

Multiplying both members of (1) by B, we obtain the integral equation

$$A = 0, \tag{2}$$

which we are to show is equivalent to equation (1).

Since A and B have no common factors, there are no values of the unknown number which make A and B zero at the same time; hence when $A = 0$, $B \neq 0$.

The necessary and sufficient condition that $\frac{A}{B} = 0$ is therefore that $A = 0$. And since $\frac{A}{B} = 0$ only when $A = 0$, equation (1) is satisfied by those values of the unknown number that make $A = 0$, that is, by the roots of (2), and by no other values.

Hence, (1) and (2) are equivalent equations, having the same roots.

PROBLEMS

212. 1. The sum of two numbers is 63, and their difference 9. What are the numbers?

Let x = the greater number.

Then $x - 9$ = the lesser number.

$\therefore x + x - 9 = 63$, the sum.

$\therefore x = 36$.

$x - 9 = 27$.

CHECK. $27 + 36 = 63$; $36 - 27 = 9$. Since errors are liable to occur in forming the equation, the substitution should always be made in the problem.

2. The sum of two numbers is s, and their difference d. What are the numbers? What, if $s = 54$ and $d = 10$?

Let x = the greater number.

Then $x - d$ = the lesser number.

$\therefore x + x - d = s.$

$$\therefore x = \frac{s + d}{2}, \tag{1}$$

and

$$x - d = \frac{s - d}{2}. \tag{2}$$

By (1), the greater number is $\frac{54 + 10}{2}$, or 32;

by (2), the lesser number is $\frac{54 - 10}{2}$, or 22.

The results obtained by solving a general problem, that is, one in which the numbers supposed to be known are represented by letters, may be used as formulæ for solving similar problems.

3. Divide the number 15 into two parts such that $\frac{1}{2}$ of the first is equal to $\frac{1}{3}$ of the second.

Let $x = \frac{1}{2}$ of the first part or $\frac{1}{3}$ of the second.

Then $2x$ = the first part,

and $3x$ = the second part.

$\therefore 2x + 3x = 15.$

$\therefore x = 3.$

$2x = 6$, the first part,

and $3x = 9$, the second part.

4. The sum of the two digits of a number is 4. If the digits are interchanged, the resulting number is equal to the original one. What is the number?

Let x = the digit in the ones' place.

Then $4 - x$ = the digit in the tens' place.

$\therefore 10(4 - x) + x$ = the original number. (Why?)

$10x + (4 - x)$ = the second number.

$\therefore 10x + (4 - x) = 10(4 - x) + x$, by the conditions.

$\therefore x = 2$, the digit in the ones' place,

and $4 - x = 2$, the digit in the tens' place.

Therefore, the original number is $10(4 - x) + x$, or 22.

5. Divide the number n into three parts such that $\frac{1}{a}$ of the first, $\frac{1}{b}$ of the second, and $\frac{1}{c}$ of the third, will be equal.

Let $x = \frac{1}{a}$ of the first part, $\frac{1}{b}$ of the second, or $\frac{1}{c}$ of the third.

Then $ax =$ the first part,

$bx =$ the second part,

and $cx =$ the third part.

$$\therefore ax + bx + cx = n.$$

$$x = \frac{n}{a+b+c},$$

$$ax = \frac{an}{a+b+c}, \text{ the first part.}$$

$$bx = \frac{bn}{a+b+c}, \text{ the second part.}$$

$$cx = \frac{cn}{a+b+c}, \text{ the third part.}$$

6. The sum of two numbers is 125, and their difference is 75. Find the numbers.

7. Divide the number 39 into two parts such that the greater shall exceed the less by 15.

8. Divide the number s into two such parts that the greater shall exceed the less by d. What are the numbers, if $s = 75$ and $d = 15$?

9. Find two consecutive numbers such that $\frac{1}{5}$ of the greater is 3 more than $\frac{1}{7}$ of the less.

10. A is twice as old as B, B is 5 years older than C, and the sum of their ages is 79 years. How old is each?

11. A horse, saddle, and bridle cost \$154. The horse cost six times as much as the saddle, and the saddle cost three times as much as the bridle. Find the cost of each.

12. The sum of the length, width, and height of a room is 45 feet. The length is $1\frac{1}{2}$ times the height, and the height is $\frac{4}{5}$ of the width. Find the dimensions.

13. One half of a certain number exceeds $\frac{1}{4}$ of it by $\frac{1}{4}$. What is the number?

14. Eight years ago Jack was twice as old as Jill. The sum of their ages now is 34 years. What was Jack's age 10 years ago?

15. What number must be added to both numerator and denominator of the fraction $\frac{5}{13}$ in order that the resulting fraction shall be $\frac{9}{7}$?

16. A number is expressed by two digits whose sum is 10. The tens' digit is 4 times the ones' digit. What is the number?

17. The sum of the three digits of a number is 10. The hundreds' digit is twice the tens' digit, and the tens' digit is 3 times the ones' digit. What is the number?

18. Find two consecutive numbers such that the fourth part and the fifth part of the less equals the third part and the ninth part of the greater.

19. Two numbers are to each other as 3 to 4. If 10 is subtracted from each, the smaller one will be $\frac{2}{3}$ of the larger. What are the numbers?

20. Two numbers are to each other as 2 to 3, and their product is 150. What are the numbers?

21. A number is composed of two digits whose sum is 8. If the digits are interchanged, the resulting number is greater by 18 than the original number. What is the number?

22. What number must be subtracted from 15 and 19 so that the differences shall be in the ratio of 5 to 7?

23. What number diminished by two thirds and one ninth of itself, and 11 more, will equal one ninth of itself?

24. Divide 80 into three parts, such that $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{5}$ of the third are equal.

25. What is the number from 5 times which if 25 is subtracted, the difference is 20 more than the remainder after taking 3 from the number?

26. At what price must goods costing \$200 be marked in order that there may be a profit of 20% after a reduction of 20% has been made?

Let x = the *number* of dollars of marked price.

Then $.20\,x$ = the *number* of dollars of discount.

Now, 20% of \$200 = \$40, gain;

\$200 + \$40 = \$240, selling price.

$\therefore x - .20\,x = 240.$

$.80\,x = 240.$

$.01\,x = 3.$

$x = 300.$

Therefore, the marked price is \$300.

CHECK. \$300 less 20% of \$300 is \$240, selling price. \$240 less 20% of \$200 is \$200, the cost price.

27. A capitalist has $\frac{2}{5}$ of his money invested at 4%, and the rest at 3%. What is his capital, if his total income is \$3600?

Let $5\,x$ = the *number* of dollars in his capital.

Then 4%, or .04, of $3\,x = .12\,x$, income at 4%;

3%, or .03, of $2\,x = .06\,x$, income at 3%.

$\therefore .12\,x + .06\,x = 3600.$ (Why?)

$x = 20{,}000,$

and $5\,x = 100{,}000.$

Therefore, his capital is \$100,000.

CHECK. $\frac{3}{5}$ of \$100,000 = \$60,000; $\frac{2}{5}$ of \$100,000 = \$40,000. 4% of \$60,000 + 3% of \$40,000 = \$3600.

28. How much water must be added to a quart of a certain medicine 20% pure so as to reduce it to a 5% solution?

Let x = the number of quarts of water to be added.

Then $.05\,(1 + x)$ = the number of quarts of medicine in reduced solution.

But .20 = the number of quarts of medicine in original solution.

$\therefore .05\,(1 + x) = .20.$

$x = 3.$

Hence 3 quarts of water must be added.

29. How much water must be added to a dekaliter of alcohol 90% pure so that the mixture shall be three fourths water?

30. In percentage the base is b; the rate, r; and the percentage, p. Find r in terms of p and b. Find b in terms of p and r.

31. Solve the interest formula, $i = prt$, for p, r, and t. Also solve the formula, $a = p + prt$, for p, r, and t.

32. A man has \$5050 invested, part of it at 4%, and the rest at 5%. The annual income is \$230. How much has he invested at each rate?

33. Divide 20 apples between A and B so that $\frac{1}{2}$ of A's number shall equal $\frac{1}{3}$ of B's.

34. A has \$150 more than B, and C has $\frac{1}{3}$ as much as A and B. They all have \$1000. How much has each?

35. I bought a lot of coffee at 12¢ a demikilo. Allowing that the coffee will fall short 5% in weighing, and that 10% of the sales will be in bad debts, for how much a demikilo must I sell it to make a clear gain of 14% on the cost?

36. In a mixture of silver and copper weighing 55 ounces, there are 15 ounces of silver. How much copper must there be added that there may be $1\frac{2}{3}$ ounces of silver in 10 ounces of the mixture?

37. A fulcrum is to be placed under a 3-foot lever so as to divide it into two parts such that 1.2 times the first shall equal 4.8 times the second. How far is it from either end?

38. A man bought 10 yards of calico and 20 yards of silk for \$30.60. The silk cost as many quarters a yard as the calico cost cents a yard. Find the price of each.

39. I bought a number of apples at the rate of 3 for a cent; sold one third of them at 2 for a cent, and the remainder at 5 for 3 cents, gaining 7 cents. How many did I buy?

40. In a certain family each son has as many brothers as sisters, but each daughter has twice as many brothers as sisters. How many children are in the family?

41. How much money must I put in a bank which allows 4% interest on deposits in order to receive $100 at the end of 9 months?

42. Sold wheat on commission at 6%, and invested the net proceeds in flour at 4% commission, my whole commission being $625. What was the value of the wheat and flour?

43. A man sold a pig and a sheep for $20.80, gaining 25% on the cost of the sheep, and 20% on that of the pig. If $\frac{2}{3}$ of the cost of the pig equaled $\frac{3}{4}$ of the cost of the sheep, what was the cost of each?

44. Sixty dollars was divided equally among a number of men. Had their number been 4 less, each would have received three times as much. How many men were there?

45. A man invested a certain sum of money in 5% stock at 80, and twice as much in 4% stock. If his income from the former stock is $300, and from the latter $1\frac{1}{3}$ times as much, what was the price of a share in the latter investment?

46. A bond dealer invests $\frac{1}{3}$ of his money in 3% bonds, $\frac{2}{5}$ of it in 4% bonds, and the remainder in 5% bonds. His income from his investment is $1180. What sum did he invest?

47. Tom and Ed find a sum of money. Tom takes $2.40 and $\frac{1}{6}$ of what was left, then Ed takes $3.52 and $\frac{1}{7}$ of what is left. They then find each has taken the same amounts. What was the sum found?

48. A field is twice as long as it is wide. By increasing its length 15 rods and its width 20 rods, the area will be increased 1070 square rods. What are the dimensions of the field?

49. At one store a lady spent $\frac{1}{2}$ her money and $\frac{1}{2}$ more; at another, $\frac{1}{2}$ of the remainder and $\frac{1}{2}$ more; at another, $\frac{1}{2}$ the remainder and $\frac{1}{2}$ more; at another, $\frac{1}{2}$ the remainder and $\frac{1}{2}$ more, and then had $1 left. How much had she at first?

50. A man invests $2000, on one part of which 12% is gained, and on the remainder 8% is lost. Find each investment, if his net gain on both was $96.

51. A son is one fourth as old as his father. Four years ago he was only one fifth as old as his father. What is the age of each?

Let x = the number of years in the father's age.

Then $\frac{x}{4}$ = the number of years in the son's age.

$x - 4$ = the number of years in the father's age 4 years ago.

$\frac{x}{4} - 4$ = the number of years in the son's age 4 years ago.

$\therefore \frac{x}{4} - 4 = \frac{1}{5}(x - 4)$.

$\therefore x = 64$, and $\frac{x}{4} = 16$.

Therefore, the father is 64 years old and the son is 16 years old.

CHECK. Give it.

52. Twelve years ago A's age was $\frac{3}{5}$ of B's age, but now his age is $\frac{5}{7}$ of B's. How old is each?

53. My wife's age plus mine equals 76 years, and $\frac{2}{3}$ of her age minus 2 years equals $\frac{1}{2}$ of my age plus 2 years. Find the age of each.

54. Ten years hence a boy will be four times as old as he was 10 years ago. How old is the boy now?

55. Tom is 5 years older than Peter, and 5 years hence he will be $3\frac{1}{2}$ times as old as Peter was 5 years ago. What is the age of each?

56. A is now 40 years of age and B 13. How many years ago was A's age 10 times the age of B?

57. A is now 25 years old and B is 15. How many years must elapse before A's age will be to B's as 5 to 4?

58. Said A to B: "I am 10 years more than twice as old as you, and in 10 years the sum of our ages will be 90 years." Find the age of each.

59. Two fifths of A's age equals $\frac{3}{4}$ of B's, and A was 21 years old when B was born. How old is each?

60. A can do a piece of work in 3 days, and B can do it in 5 days. How long will it take both together?

Let x = the number of days required.

Then $\frac{1}{x}$ = the part both can do in 1 day.

Now, $\frac{1}{3}$ = the part A can do in 1 day,

and $\frac{1}{5}$ = the part B can do in 1 day.

$\therefore \frac{1}{3} + \frac{1}{5}$ = the part both can do in 1 day.

$$\therefore \frac{1}{3} + \frac{1}{5} = \frac{1}{x},$$

$$5x + 3x = 15.$$

$$x = 1\tfrac{7}{8}.$$

Hence they can do the work together in $1\frac{7}{8}$ days.

61. A can do a piece of work in a days, and B can do it in b days. In how many days can both together do it?

Let x = the number of days.

Then $\frac{1}{x}$ = the part both can do in 1 day,

$\frac{1}{a}$ = the part A can do in 1 day,

and $\frac{1}{b}$ = the part B can do in 1 day.

$$\therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$$

Clearing of fractions, $bx + ax = ab.$

$$\therefore x = \frac{ab}{a+b}.$$

Hence they can do the work together in $\frac{ab}{a+b}$ days.

62. A and B can do a piece of work in 6 days, and A can do it in 9 days. In how many days can B do the work?

63. A water tank is supplied by 3 pipes. The first will fill it in 2 hours, the second in 3 hours, and the third in 5 hours. How long will it take all three together to fill it?

64. An empty cistern has 2 pipes. One fills it in 40 minutes, the other empties it in 60 minutes. If both are opened, in what time will the cistern be filled?

65. A can mow an acre of grass in $\frac{2}{3}$ of a day, and B in $\frac{2}{5}$ of a day. How many acres can both mow in 2 days?

66. A can do a piece of work in a days; A and B together can do it in b days. In how many days can B do it alone?

67. A circular pond contains 314.16 square yards. What is its diameter?

68. At what time between 2 and 3 o'clock will the hour and minute hands of a clock be together?

At 2 o'clock the hands are 10 minute-spaces apart.

Let $x =$ the number of spaces moved by the minute hand.

Then $x - 10 =$ the number of spaces moved by the hour hand.

Now, since the minute hand moves 12 times as fast as the hour hand, we have the equation, $12(x - 10) = x$;

whence $x = 10\frac{10}{11}$.

Hence the time is $10\frac{10}{11}$ minutes past 2 o'clock.

69. At what time between 4 and 5 o'clock will the hour and minute hands of a clock be at right angles to each other?

To be at right angles they must be 15 minute-spaces apart. At 4 o'clock they are 20 spaces apart; hence to be at right angles the minute hand must gain $20 - 15$ or $20 + 15$ such spaces, that is, 5 or 35 spaces.

Let $x =$ the number of spaces moved by the minute hand.

Then $x - 5$ or $x - 35 =$ the number of spaces moved by the hour hand.

$\therefore\ 12(x - 5)$ or $12(x - 35) = x$;

whence $x = 5\frac{5}{11}$, or $38\frac{2}{11}$.

Hence the time is $5\frac{5}{11}$ minutes or $38\frac{2}{11}$ minutes past 4 o'clock.

70. A courier who goes at the rate of b miles an hour is followed, after he has gone m miles, by another who goes at the rate of a miles an hour. In how many hours after the second starts will he overtake the first, if $a > b$?

Let $x =$ the number of hours required.

Then $ax =$ the number of miles traveled by the second courier,

and $bx =$ the number of miles traveled by the first courier.

$\therefore\ ax - bx = m$,

$x = \dfrac{m}{a - b}$, the number of hours required to overtake the first courier.

71. At what time between 4 and 5 o'clock are the hands of a clock in the same straight line in opposite directions? At what time between 7 and 8 o'clock?

72. At 2 P.M. a train running 30 miles an hour leaves Pittsburg for Chicago, and at 5 P.M. another train running 40 miles an hour leaves Pittsburg for the same destination. In how many hours will the second train overtake the first?

73. At what time between 10 and 11 o'clock is the minute hand of a clock 25 minutes in advance of the hour hand?

74. At what time after midnight are the hour and minute hands of a clock first together?

75. At what time between 5 and 6 o'clock is the hour hand of a clock as many minute spaces after 5 o'clock as the minute hand is spaces before 10 o'clock?

76. At what time between 1 and 2 o'clock are the hour and minute hands of a clock together? Between 3 and 4? 7 and 8? 11 and 12?

77. At what time between 10 and 11 o'clock do the hour and minute hands of a watch coincide?

78. At what time between 1 and 2 o'clock are the hands of a clock exactly opposite each other?

79. A and B start from the same point and travel in the same direction. A goes 7 kilometers an hour and B 3 kilometers an hour. If B has a start of 5 hours, when will he be overtaken by A?

---

80. A bicyclist, riding at the rate of 8 miles an hour, after a start of 18 miles is followed by another at the rate of 10 miles an hour. After how many hours will the faster rider overtake the slower? How far will each have traveled?

81. An automobile, moving at the rate of r miles an hour, had gone p miles when another started in pursuit at the rate of n miles an hour. In what time did the second overtake the first? In what time, if $r = 18$, $p = 54$, $n = 27$?

82. A man rowed upstream at the rate of a miles an hour and returned at the rate of b miles an hour. He was gone m hours. How far upstream did he row? How far, if $a = 3$, $b = 4\frac{1}{2}$, $m = 10$?

83. A and B can do a piece of work in a days; B and C in b days; C and D in c days; A and D in d days. In how many days can all working together do it?

84. A bicyclist going at the rate of $10\frac{1}{2}$ miles an hour punctured a tire and walked back at the rate of $3\frac{3}{4}$ miles an hour. He was gone $4\frac{1}{2}$ hours. How far did he travel in all?

85. A man rows downstream at the rate of a kilometers an hour and returns at the rate of b kilometers an hour. How far downstream can he go and return in c hours? How far, if $a = 6$, $b = 4$, and $c = 5$?

86. A man engaged to work for a days on condition that he was to receive b dollars for each day he worked, and forfeit c dollars for each day he was idle. At the end of the time he received d dollars. How many days had he worked? How many, if $a = 6$, $b = 5$, $c = 3$, and $d = 20$?

87. An officer, attempting to arrange his men in a solid square, found that with a certain number of men on a side he had 31 men left over. He then added 2 men to each side of his square and found that he lacked 25 men of enough to complete this square. How many men had he?

88. How many ounces of copper must be melted with 10 oz. of gold 18 carats fine ($\frac{18}{24}$ pure) to make an ingot 14 carats fine?

89. A farmer agreed to give his hired man \$100 and 2 cows for a year's labor. The man quit at the end of 10 months, receiving the cows and \$70 as a fair settlement. At how much were the cows valued?

90. A man rides from his station to a certain city on a train going at the rate of 30 miles an hour, and returns from the city at the rate of 28 miles an hour. Arriving at his station on his return he found that he had been gone 16 hours, including $1\frac{1}{2}$

hours' stay in the city. Find the distance of the city from his station.

91. What number must be added to the numerator and subtracted from the denominator of the fraction $\frac{7}{16}$, to give its reciprocal, $\frac{16}{7}$?

92. The denominator of a fraction exceeds its numerator by 3, and if 1 be added to both numerator and denominator, the resulting fraction will be equal to $\frac{5}{8}$. What is the fraction?

93. A dealer bought some hogs at \$3 a head. Had he bought 2 less for the same money, they would have cost him \$1 more a head. How many hogs did he buy?

94. If the first of the two digits of a number be doubled, it will be 3 more than the second, and the number itself is 6 less than 5 times the sum of the digits. What is the number?

95. Divide n into two parts such that one divided by the other will give q as a quotient and r as a remainder. What is the result, if $n = 313$, $q = 2$, and $r = 19$?

96. A garrison of 1200 men had provisions to last 90 days, but 30 days later it was reënforced, and from that time the provisions lasted only $4\frac{1}{2}$ days. What was the number of the reënforcement?

97. How many pounds of fresh water must be added to 80 pounds of sea water containing by weight 5% of salt, in order that 45 pounds of the new solution shall contain only $3\frac{1}{3}$% of salt?

98. The area of a rectangle is 5 square rods greater than that of a square. How long is the side of the square, if it is 5 rods shorter than the longer and 3 rods greater than the shorter side of the rectangle?

99. A merchant adds yearly to his capital $\frac{1}{3}$ of it, but takes from it at the end of each year \$4000 for expenses. At the end of the third year, after deducting the last \$4000, he has twice his original capital. What was his original capital?

100. A train leaves P for Q at the rate of 25 miles an hour. One hour later a train leaves M and runs through P to Q at the rate of 35 miles an hour, arriving at Q 24 minutes later than the first train. The distance from M to P being 21 miles, find the distance from P to Q.

101. What is the time of the day when $\frac{2}{3}$ of the time to noon is equal to $\frac{2}{5}$ of the time past midnight?

102. At the time of marriage a wife's age was $\frac{3}{5}$ of the age of her husband, and 12 years after marriage her age was $\frac{5}{7}$ of the age of her husband. Find the age of each at marriage.

103. Divide the number m into 4 parts such that the first increased by n, the second diminished by n, the third multiplied by n, and the fourth divided by n will all be equal. Find the results, if $m = 90$ and $n = 2$.

104. In a shooting match one marksman hit the bull's-eye 8 times out of 12, and another hit it 10 times out of 12. Both together hit the bull's-eye 36 times, each firing the same number of shots at the target. Find the number of shots fired by each.

105. A man who has a certain sum of money to invest finds that, by buying 5% stock at 90, he will receive \$ 10 more income than if he buys 8% stock at 150. What sum has he to invest?

106. If 194 grams of gold weighs 184 grams in water, and 63 grams of silver weighs 57 grams in water, how many grams of gold and silver are there in a mass of gold and silver weighing 160 grams in air and 149 grams in water?

107. In how many years will m dollars amount to n dollars at r per cent interest?

108. At what rate of simple interest will a dollars amount to b dollars in t years?

109. A man rode from A to B on a train, and back on a wheel; he was gone q hours. If the train's rate was r miles an hour and the wheel's rate p miles an hour, what was the distance from A to B?

110. If a pounds of coffee are worth a dollar, and b pounds of sugar are worth the same, and c pounds of both are worth a dollar, how many pounds of coffee are in the c pounds of mixture?

111. A man wishing to buy a flock of sheep found that he would need m dollars more than he had if he paid a dollars each, but that if he paid b dollars each he would have n dollars left. How many sheep were there? How many, if $m = 10$, $a = 5$, $b = 3$, $n = 50$?

112. The distance from A to B is m miles. Tom left A at 1 P.M. to meet Harry, who left B at 1.30 P.M. Tom walked p miles an hour, Harry q miles. At what hour did they meet?

113. A farm is rented for \$ 300 in cash and a certain number of bushels of wheat. When wheat is 80¢ a bushel the rent is $12\frac{1}{2}\%$ lower than when wheat is \$ 1.20 a bushel. Find the number of bushels of wheat given for rent.

114. An officer with a warrant for a criminal starts at a rate which will require 6 hours to serve the paper. When halfway there the warrant is given to another officer, who rides 3 miles an hour faster than the first, and is therefore served $\frac{1}{2}$ hour sooner than it would have been by the first. At what rate did the first officer travel?

DISCUSSION OF THE SOLUTIONS OF PROBLEMS

213. In solving an *equation* the result is mathematically correct if the values obtained for the unknown number satisfy the equation, but if the equation has arisen in connection with any particular *problem*, some of the results, or all of them, may be excluded by the restrictions imposed. For this reason it is important before proceeding further to emphasize the meaning of positive, negative, and zero results as we find them illustrated in the solution of the problems in which they occur.

The discussion of indeterminate and infinite results is postponed.

Positive Results

214. In room A there are 87 pupils and in room B 40 pupils. How many pupils must be transferred from room A to room B in order that there shall be twice as many pupils in the former as in the latter?

Let x = the number of pupils to be transferred.

Then $87 - x$ = the number of pupils left in room A;

also $40 + x$ = the number in B after the transfer.

$\therefore\ 87 - x = 2(40 + x)$, by the conditions.

$\therefore\ -3x = -7,$

$x = 2\frac{1}{3}.$

The result $2\frac{1}{3}$ satisfies the *equation*, but not the *problem*, for the number of pupils transferred cannot be fractional. That is, the practical solution of the problem is *impossible* by reason of the physical conditions imposed.

Negative Results

215. A father is 45 years old and his son 15. After how many years will the father be 4 times as old as the son? How many years ago was he 4 times as old?

Let x = the required number of years.

Then $45 + x = 4(15 + x)$, by the conditions.

Solving, $x = -5.$

The value $x = -5$ satisfies the equation, but in an arithmetical sense it is absurd. From the nature of the problem the conditions expressed are inconsistent, since at no time in the *future* will the father be 4 times as old as the son. The difficulty presented by this apparently meaningless result may be met in either of two ways:

(1) Because of the relation of positive and negative numbers, we may interpret "-5 years after" to mean "5 years before"; or

(2) Since to add -5 is equivalent to subtracting $+5$, we may restate the problem so that the question shall read, "How many years ago," as in the second query. For the latter question the equation is

$$45 - x = 4(15 - x);\ \text{whence}\ x = 5.$$

The *absolute value* of the result is the same in either case.

Zero Results

216. 1. A father is 45 years old and his son 15. After how many years will the father be 3 times as old as the son?

Let x = the required number of years.

Then $45 + x = 3(15 + x)$, by the conditions.

$\therefore\ 2x = 0,$

and $x = \frac{0}{2} = 0.$

This result is true, for the father is *now* 3 times as old as the son.

2. Having two kinds of coffee worth 25 and 20 cents a pound, respectively, how many pounds of each must be taken to make a mixture of 100 pounds worth \$ 20?

Let x = the number of pounds of the better kind.

Then $100 - x$ = the number of pounds of the cheaper kind.

$\therefore\ \frac{1}{4}x + \frac{1}{5}(100 - x) = 20,$

whence $x = 0.$

This result means that no such mixture can be made, for it is evident from the problem that 100 pounds of the poorer kind are worth \$20.

In 1, zero is the proper answer; in 2, it indicates that the problem is *impossible*. In both cases the value $x = 0$ satisfies the equation.

Solve the following problems and discuss the results:

3. A is 25 years old and B is 15 years older. In how many years will B be twice as old as A?

4. In one room there are 95 persons and in another room 23. How many persons must pass from the first room into the second in order that there shall be 3 times as many persons in the first room as in the second?

5. A father is a years old and his son is b. After how many years will the father be c times as old as the son? State under what conditions the result will be positive, negative, or zero.

6. A dealer has two kinds of tea worth a and b dollars a pound, respectively. How many pounds of each must he take to make a mixture of n pounds worth c dollars? Interpret for $n \times b = c$.

7. A bicyclist passes through P at the rate of r miles an hour, and t hours later another passes in pursuit at the rate of p miles an hour. How far from P will the second rider overtake the first? What will be the distance, if $r = 10$, $p = 12$, and $t = 8$? Find and interpret the result, if $r = 12$, $p = 10$. Restate the problem so that the negative result will apply.

SIMPLE EQUATIONS

WITH TWO OR MORE UNKNOWN NUMBERS

217. The single linear equation $x+y=10$ expresses a *relation* between two unknown numbers; it states merely that their sum is 10. From it we find $y=10-x$. If $x=1$, $y=9$; if $x=2$, $y=8$; if $x=-2$, $y=12$; and so on. If $x=7$, what is the value of y? If $x=-8$? If $x=-20$? If $x=50$? By assuming various values for x, can we find corresponding values for y?

218. When a single linear equation contains two or more unknown numbers, it will be satisfied by an indefinite number of pairs of values for the unknown numbers. An equation of this kind is therefore said to be *indeterminate.*

Thus, the equation $x+y=5$ does not give determinate values for x and y, because any number of pairs of values may be found that will satisfy the equation. If only one pair of values would satisfy the equation, it would be determinate.

219. The two linear equations $x+y=10$ and $x-y=2$ express *two relations* between the two unknown numbers: one states that their sum is 10, the other that their difference is 2. As we have seen, an indefinite number of values of x and y will satisfy either equation alone, but only one pair of values will satisfy both equations.

Thus, in $x+y=10$, if $x=3$, $y=7$; but, using these values in $x-y=2$, we have $3-7=2$, which is not true. To satisfy both equations, x must have the value 6, and y the value 4.

220. When we have *two* linear equations containing *two* unknown numbers, there is, in general, but one pair of values that will satisfy *both* equations, although an unlimited number of pairs will satisfy either equation *alone*.

In like manner, if we have a system of *three* linear equations containing *three* unknown numbers, we may, in general, find determinate values for the three unknown numbers; and so on for systems of equations containing more than three unknown numbers.

221. Simultaneous Equations.

Two or more equations are said to be **simultaneous** when the values of the unknown numbers are the same in each of them.

Thus, $x+y=4$ and $x-y=6$ are simultaneous equations, for each is satisfied by the values $x=5$ and $y=-1$. But $x-y=4$ and $x-y=6$ are not simultaneous, because they are not satisfied by the same values of x and y; they are said to be *inconsistent*.

222. When equations express *different* relations between the unknown numbers, they are said to be *independent*.

Thus, $x+y=6$ and $x-y=2$ are independent equations; they express different relations between x and y, and cannot be reduced to the same form. The equations $x+y=5$ and $2x+2y=10$ are simultaneous; but the second can be reduced to the form of the first, its members being merely twice those of the first. Hence, the equations are not independent.

223. The equations $x+y=2$ and $x+y=1$ have evidently no common pair of roots which will satisfy each, since we would then have $2=1$. They are *inconsistent*.

Also, the equations $x+y=5$ and $3x+3y=15$ have no determinate solution, for the second is derived from the first by multiplying both members of the former by 3. The two equations are *equivalent* and reduce to a *single indeterminate equation*.

Hence, it is seen that in order to furnish a determinate solution, the equations must be both *simultaneous* and *independent*.

224. Two sets of simultaneous equations, each set having the same roots as the other, are called **Equivalent Systems** of equations.

Thus, $\left\{\begin{matrix} 3x+2y=44 \\ x+\ y=17 \end{matrix}\right\}$ and $\left\{\begin{matrix} 3x+2y=44 \\ 2x+2y=34 \end{matrix}\right\}$

are equivalent systems, for each is satisfied by $x=10$, $y=7$, and by no other values.

225. Two simultaneous equations containing two unknown numbers are solved by combining the equations so as to obtain a single equation with but one of the unknown numbers. The unknown number that is caused to disappear in the process is said to be *eliminated.*

226. There are three principal **Methods of Elimination**: (1) By addition or subtraction, (2) By substitution, (3) By comparison. These methods are all closely related, as will be seen from a comparison of the illustrative solutions which follow.

227. Elimination by Addition or Subtraction.

Elimination by addition or subtraction consists in multiplying both members of each of the equations by some number which will make the coefficients of one of the unknown numbers the same in both, and in adding or subtracting the corresponding members of the resulting equations according as the equal terms have unlike or like signs in the two equations. This is the method usually employed, the other methods being the easier only in special cases.

1. Solve the system of equations

$$\left\{\begin{matrix} 5x+3y=68. & (1) \\ 3x-2y=18. & (2) \end{matrix}\right.$$

To cause one of the unknown numbers to disappear, we must give either the x's or the y's the same coefficient. This is conveniently effected by multiplying both members of (1) by 2, and of (2) by 3.

Multiplying (1) by 2, $\quad 10x+6y=136.$ (3)

Multiplying (2) by 3, $\quad 9x-6y=54.$ (4)

Adding (3) and (4), member by member, $19x = 190.$

$\therefore x = 10.$

Substituting this value of x in (1), $10 \times 5 + 3y = 68.$

$\therefore 3y = 18,$

and $y = 6.$

After finding the value of x, from which equation did we obtain the value of y? Is that any reason for using the *other* equation in checking? Why?

What multipliers were used in eliminating y? What multipliers would be used to eliminate x? Is it more convenient to multiply by small numbers than by larger ones?

2. Solve the system of equations

$$\begin{cases} 2x - 3y = -4. & (1) \\ 6x - 2y = 2. & (2) \end{cases}$$

Multiplying (1) by 2, $4x - 6y = -8.$ (3)

Multiplying (2) by 3, $18x - 6y = 6.$ (4)

Subtracting (4) from (3), $-14x = -14.$

$\therefore x = 1.$

Substituting this value in (1), $2 - 3y = -4.$

$\therefore -3y = -6,$

and $y = 2.$

In the solution of Ex. 1 y is eliminated by *addition*. How is it eliminated in this solution? Could x have been eliminated first by subtraction?

Multiply the first equation by -2 and the second equation by $+3$, and eliminate y by *adding* the resulting equations. How, then, may elimination be effected by *addition* in all cases? Is subtraction merely a special case under that of addition?

3. Solve the system of equations

$$\begin{cases} 2x + 5y = 11. & (1) \\ x - y = 2. & (2) \end{cases}$$

Multiplying (2) by 2, $2x - 2y = 4.$ (3)

Multiplying (1) by 1, $2x + 5y = 11.$ (4)

Subtracting (3) from (4), $7y = 7,$

whence $y = 1.$

Substituting this value in (1), $2x + 5 = 11,$

whence $x = 3.$

It sometimes happens, as in this example, that but one equation needs to be multiplied. The one that remains unchanged is equivalent to itself multiplied by 1.

4. Solve the system of equations

$$\begin{cases} 4x - 8y = 4. & (1) \\ x + y = 7. & (2) \end{cases}$$

Dividing (1) by 4, $x - 2y = 1.$ (3)

$x + y = 7.$ (2)

Subtracting (3) from (2), $3y = 6,$

whence $y = 2.$

We may find the value of x by substituting this value of y in (1) or (2). We may also find it by eliminating y from (1) and (2). Find it in both ways.

Is dividing (1) by 4 equivalent to multiplying it by $\frac{1}{4}$? Which seems easier?

5. Solve the system of equations

$$\begin{cases} \frac{2}{x} - \frac{3}{y} = 5. & (1) \\ \frac{5}{x} - \frac{2}{y} = 7. & (2) \end{cases}$$

Multiplying (1) by 2, $\frac{4}{x} - \frac{6}{y} = 10.$ (3)

Multiplying (2) by 3, $\frac{15}{x} - \frac{6}{y} = 21.$ (4)

Subtracting (3) from (4), $\frac{11}{x} = 11.$

$\therefore \frac{1}{x} = 1,$

and $x = 1.$

Substituting this value in (2), $\frac{5}{1} - \frac{2}{y} = 7.$

$\therefore -\frac{2}{y} = 2,$

and $y = -1.$

Check the results. Why should these equations not be cleared of fractions at once? Would clearing give a term xy?

6. Solve the system of equations

$$\begin{cases} ax + by = m. & (1) \\ cx + dy = n. & (2) \end{cases}$$

Multiplying (1) by d, $adx + bdy = dm.$ (3)

Multiplying (2) by b, $bcx + bdy = bn.$ (4)

Subtracting (4) from (3), $(ad - bc)x = dm - bn.$

$$\therefore x = \frac{dm - bn}{ad - bc}.$$

Multiplying (1) by c, $acx + bcy = cm.$ (5)

Multiplying (2) by a, $acx + ady = an.$ (6)

Subtracting (5) from (6), $(ad - bc)y = an - cm.$

$$\therefore y = \frac{an - cm}{ad - bc}.$$

In this equation it is easier to find the value of y by eliminating x than by substituting the value found for x in one of the given equations. In literal equations this method is usually easier.

Solve the following systems of equations by addition or subtraction:

7. $5x + 3y = 26.$
$3x - y = 10.$

8. $7x + 2y = 47.$
$5x - 4y = 1.$

9. $2x - 9y = 11.$
$3x - 4y = 7.$

10. $3x + 4y = 25.$
$4x + 3y = 31.$

11. $6x + 7y = 29.$
$4x - 3y = 4.$

12. $2x + 3y = 19.$
$3x + 2y = 16.$

13. $-5x + 6y = 18.$
$12x - 9y = 0.$

14. $3x - 11y = 0.$
$19x - 19y = 8.$

15. $4x - 5y = 5.$
$3x + 2y = 21.$

16. $5x - 9y = 1.$
$8x - 10y = -5.$

17. $3y + 7z = 50.$
$5y - 2z = 15.$

18. $8x + 3y = 3.$
$12x + 9y = 3.$

19. $x+y=a.$
$x-y=b.$

20. $ax+by=2ab.$
$bx-ay=b^2-a^2.$

21. $\frac{1}{2}x+\frac{1}{3}y=13.$
$\frac{1}{5}x+\frac{1}{8}y=5.$

22. $\frac{x-4}{4}+\frac{y-3}{3}=3.$
$\frac{x-3}{3}-\frac{y-4}{4}=2\frac{1}{2}.$

23. $\frac{x}{3}-\frac{y}{2}=2.$
$\frac{x}{2}+\frac{y}{4}=7.$

24. $\frac{1}{x}-\frac{1}{y}=1.$
$2x-y=7xy.$

25. $y(x+7)=x(y+1).$
$2y+20=3x+1.$

26. $3x+\frac{2}{3}y=28.$
$5x-\frac{1}{2}y=37.$

27. $0.7x-0.02y=2.$
$0.7x+0.02y=2.2.$

28. $\frac{3}{8}y-\frac{1}{5}x=24.$
$\frac{2}{7}y+\frac{1}{3}x=-11.$

29. $\frac{x+3}{y+3}=\frac{5}{6}.$
$\frac{x-3}{y-3}=\frac{2}{3}.$

30. $\frac{x}{m}+\frac{y}{n}=1.$
$\frac{x}{m}-\frac{y}{n}=\frac{1}{2}.$

31. $\frac{2x}{3}+\frac{y}{4}=1.$
$12x+5y=16.$

32. $6x+5y=540.$
$4x-2y=168.$

228. Elimination by Substitution.

Elimination by substitution consists in finding the value of one of the unknown numbers from one of the equations, and then substituting this value in the other equation. This method is more convenient in special cases, especially when the coefficient of either unknown number is 1.

1. Solve the system of equations

$$\begin{cases} 3x+4y=18. & (1) \\ x+2y=8. & (2) \end{cases}$$

Solving equation (2) for x, we have

$$x=8-2y.$$

Substituting this value of x in (1),

$$3(8 - 2y) + 4y = 18,$$

or $$24 - 6y + 4y = 18.$$

$$\therefore\ -2y = -6,$$

and $$y = 3.$$

Substituting this value in (2),

$$x + 6 = 8.$$

$$\therefore\ x = 2.$$

In this solution x is eliminated by *substitution*. Observe that the coefficient of x in equation (2) is 1. Why is this a convenient method when the coefficient of x or y is 1? Is it necessary that this be the case in order that this method may apply?

229. Elimination by Comparison.

Elimination by comparison consists in finding the value of one of the unknown numbers in each of the two equations, and placing these values equal to each other. This method is least used, and is merely a special case under that of substitution, as will appear by considering the solution below.

1. Solve the system of equations

$$\begin{cases} x + 4y = 1. & (1) \\ x - 5y = 19. & (2) \end{cases}$$

Solving both (1) and (2) for x, we have

$$x = 1 - 4y, \quad (3)$$

and $$x = 19 + 5y. \quad (4)$$

Comparing (equating) these values of x, we have

$$1 - 4y = 19 + 5y. \quad \text{(Ax. 1)}$$

$$\therefore\ -9y = 18,$$

and $$y = -2.$$

By substitution in (3), $$x = 9.$$

Observe that *comparing* the values of x in (3) and (4) is the same thing as *substituting* the value of x from (3) in (4). Point out the advantages of this method when the coefficients of x, or of y, are both 1. Will any one of the three methods solve a system of equations? Why do we substitute the value of x in (3) in preference to (4)?

Solve the following systems of equations by substitution or comparison:

2. $x - 2y = 3.$
$2x - y = 27.$

3. $x - y = 3.$
$x + 4y = 13.$

4. $3x - 4y = 18.$
$3x + 2y = 0.$

5. $x - 12y = 3.$
$x + 4y = 19.$

6. $2x - y = 5.$
$5x - 2y = 14.$

7. $7y - 3x = 18.$
$2y - 5x = 1.$

8. $x + 5y = 20.$
$x - 7y = -16.$

9. $8x - y = -6.$
$x + 8y = -17.$

10. $3x - y = 73.$
$2x - y = 32.$

11. $4x - 7y = 19.$
$4x + 9y = 67.$

12. $x - 2y = 25.$
$3x - 5y = 66.$

13. $5x + 2y = 47.$
$2x - y = 8.$

14. $y - 2z = -6.$
$y + 2z = 34.$

15. $x + 8y = 17.$
$7x - 3y = 1.$

16. $4x + 7y = 29.$
$x + 3y = 11.$

17. $x - 3y = -19.$
$y - 3x = -23.$

18. $2x - 6y = -38.$
$y - 3x = -23.$

19. $10x - y = 9.$
$10y - x = 108.$

20. $3y - 4x = 1.$
$3x - 2y = 1.$

21. $x - 6y = 2.$
$3y - 8x = 29.$

22. $ax + by = a^2.$
$bx + ay = b^2.$

23. $ax - by = 10.$
$ax + by = 26.$

24. $y - 3x = 9.$
$2x + 7y = -6.$

25. $x = 16 - 4y.$
$y = 34 - 4x.$

26. $x + y = a.$
$x - by = 0.$

27. $cx = dy.$
$x + y = c.$

28. $\frac{x}{5}+\frac{y}{2}=5.$

$\frac{x}{2}-\frac{2y}{5}=5.$

29. $\frac{5}{x}+\frac{3}{y}=30.$

$\frac{9}{x}-\frac{5}{y}=2.$

230. The following general directions will serve to some extent to train the student's judgment to choose the best method of attack in solving sets of equations; but he must carefully consider each problem in order to determine which method is best in that particular example.

(1) *It is generally best to eliminate by addition or subtraction, especially in solving literal equations.*

(2) *If* 1 *is the coefficient of either or both unknown numbers, it is often preferable to eliminate by substitution or comparison.*

(3) *If the equations are in fractional form, it is usually best to eliminate without clearing. But in some cases it is better to clear of fractions at once, reducing the resulting equations to the simplest form before eliminating.*

Solve the following systems of equations, checking results:

1. $6x-5y=7.$

$15x-7y=23.$

2. $3x+2y=8.$

$4x-3y=5.$

3. $2.7x+2.6y=8.8.$

$0.9x+2.2y=4.4.$

4. $\frac{x+y}{3}+x=15.$

$\frac{x-y}{5}+y=6.$

5. $\frac{1}{x-1}-\frac{3}{x+y}=0.$

$\frac{3}{x-y}+3=0.$

6. $\frac{2x+4}{3y-1}=1.$

$\frac{7x-2}{y+3}=2.$

7. $\frac{x}{a}+\frac{y}{b}=1.$

$\frac{x}{b}-\frac{y}{a-b}=1.$

8. $\frac{2}{3}x+3y=42.$

$3x-\frac{4}{5}y=46.$

9. $\frac{m}{x}+\frac{n}{y}=a.$

$\frac{r}{x}+\frac{s}{y}=b.$

10. $x + \frac{1}{8}y = 11.$
$\frac{1}{8}x + 3y = 21.$

11. $\frac{3}{x-1} + \frac{4}{y-1} = 0.$
$\frac{5}{2x-3} - \frac{7}{2y+13} = 0.$

12. $\frac{x}{a} + \frac{y}{b} = c.$
$ax + bcy = d.$

13. $\frac{x}{a} + \frac{y}{b} = c.$
$\frac{x}{b} + \frac{y}{c} = d.$

14. $\frac{x+y}{8} + \frac{x-y}{6} = 5.$
$\frac{x+y}{4} - \frac{x-y}{3} = 10.$

15. $\frac{x}{c} + \frac{y}{d} = 1.$
$x - y = c - d.$

16. $3y - 2x = -1.$
$5y - 4x = -4.$

17. $3x + 2y = 118.$
$x + 5y = 191.$

18. $\frac{a}{x} + \frac{b}{y} = c.$
$\frac{b}{x} + \frac{c}{y} = d.$

19. $\frac{1}{x-1} + \frac{2}{y+1} = \frac{5}{6}.$
$\frac{3}{x-1} + \frac{4}{y+1} = 2.$

20. $3ax + 2by = c.$
$a^2x + b^2y = 5bc.$

21. $\frac{1}{x} - \frac{1}{y} = m.$
$\frac{1}{x} + \frac{1}{y} = n.$

22. $x + y = a + b.$
$ax + by = a^2 + b^2.$

23. $\frac{m}{nx} + \frac{n}{my} = m + n.$
$\frac{n}{x} + \frac{m}{y} = m^2 + n^2.$

24. $\frac{x}{m+n} - \frac{y}{m-n} = 1.$
$\frac{x}{m+n} + \frac{y}{m-n} = 10.$

25. $m(x+y) - n(x-y) = 2m.$
$m(x-y) - n(x+y) = 2n.$

26. $(a+b)x - (a-b)y = 4ab.$
$(a-b)x + (a+b)y = 0.$

27. $2x - \frac{y-3}{5} = 4.$
$3y + \frac{x-2}{3} = 9.$

28. $\frac{1}{2}x + \frac{1}{3}y = \frac{3}{4}.$
$\frac{1}{3}x - \frac{1}{4}y = \frac{1}{6}.$

29. $\frac{x+a}{y-b} = \frac{b}{a}.$
$x + y = 2b.$

30. $\frac{cx}{d} - \frac{dy}{c} = \frac{c^3 - d^3}{cd}.$
$cx - dy = c^2 - d^2.$

31. $\dfrac{x-y+1}{x+y-1}=a.$

$\dfrac{x+y+1}{x-y-1}=b.$

32. $ax+\dfrac{b}{y}=1.$

$bx+\dfrac{a}{y}=1.$

33. $\dfrac{m}{m+x}+\dfrac{n}{n-y}=\dfrac{m}{n}.$

$\dfrac{n}{m+x}-\dfrac{m}{n-y}=\dfrac{n}{m}.$

34. $\dfrac{x}{a+b}-\dfrac{y}{a-b}=\dfrac{1}{a-b}.$

$\dfrac{x}{a-b}+\dfrac{y}{a+b}=\dfrac{1}{a+b}.$

EQUIVALENT SYSTEMS OF EQUATIONS

231. The principles of equivalence of equations, which have been proved for equations of one unknown number (Art. 211), apply to any equation of a system. We shall proceed to a proof of the principle, which has hitherto been assumed, that extraneous roots are not introduced in combining simultaneous equations in the process of elimination. This principle is stated in the following

THEOREM. **If any equation of a system is replaced by an equation obtained by taking the sum or difference of a times that equation and b times another equation of the system, the resulting equation is equivalent to the given one.**

The equations of any system can be transformed so that the second member of each is 0, by the principle stated in Art. 211, Th. I.

Let
$$\left.\begin{aligned} A&=0,\\ B&=0,\end{aligned}\right\} \tag{1}$$

be any given system containing x and y, and let a and b be two *known* numbers, not zero, and either positive or negative. Then we are to show that

$$\left.\begin{aligned} aA+bB&=0,\\ B&=0,\end{aligned}\right\} \tag{2}$$

is an equivalent system.

A solution of system (1) gives values of the unknown numbers that make $A = 0$ and $B = 0$; hence, since a and b are any known numbers, except zero, it follows that any pair of values that makes $A = 0$ makes $aA = 0$, and any pair that makes $B = 0$ makes $bB = 0$, Art. 211, Th. II. Therefore $aA + bB = 0$.

Hence the same pair of values that satisfies system (1) also satisfies system (2), since $B = 0$ is common to the two systems.

Also, a solution of system (2) gives values of the unknown numbers that make B, or $bB = 0$ and $aA + bB = 0$; hence make $aA = 0$. And since a is not 0, $A = 0$.

Therefore the same pair of values that satisfies system (2) also satisfies system (1).

Since all the roots of system (1) are roots of system (2), and all the roots of system (2) are roots of system (1), it follows that the two systems are equivalent.

Similarly the Theorem can be proved for a system of any number of equations containing the same number of unknown numbers.

PROBLEMS

232. The solution of problems involves more than the solving of equations. The student must first *form* the equations from the conditions contained in the problem. In his endeavor to secure certain information required by the problem, he must, through the language employed, clearly see the conditions expressed or implied, and by the use of his knowledge and reasoning power must translate these conditions into algebraic statements; that is, into equations.

In order that a problem may be solved, the number of conditions it contains must equal that of the required numbers. In the problem, "If 3 times A's age is 60 years, how old is he?" there is but one number required and but one condition; hence this can be solved by means of a single equation, but not by means of a system of two or more equations. In the problem, "A and B together have \$18, and A has \$2 more than B,"

there are two required numbers and two conditions; hence it can be solved by means of a system of two equations.

Let x = the number of dollars A has,

and y = the number B has.

First condition: A's number + B's number = 18.

Algebraic statement: $x + y = 18.$ (1)

Second condition: A's number = B's number + 2.

Algebraic statement: $x = y + 2,$ (a)

$x - 2 = y,$ (b) (2)

or $x - y = 2.$ (c)

The condition that "A has \$2 more than B" is expressed in equation (a), and *implies* that B has \$2 less than A, which is stated in (b), or that the difference between their sums is \$2, which is stated in (c). Either (a), (b), or (c) may be taken as the second equation of the system.

1. A certain number represented by two figures is equal to 5 times the sum of its digits, and when 9 is added to the number, the order of the digits is reversed. What is the number?

Let x = the tens' digit,

and y = the ones' digit.

Then $10x + y$ = the number,

and $10y + x$ = the number with digits reversed.

Then by the first condition,

$$10x + y = 5(x + y),$$

or $$5x - 4y = 0. \quad (1)$$

By the second condition,

$$10x + y + 9 = 10y + x,$$

or $$9x - 9y = -9,$$

or $$x - y = -1. \quad (2)$$

Multiplying (2) by 4, $4x - 4y = -4.$ (3)

Subtracting (3) from (1), $x = 4.$

Substituting this value in (2),

$$4 - y = -1.$$

$$\therefore y = 5.$$

Hence, $10x + y = 40 + 5$, or 45, the number.

2. What is that fraction which equals $\frac{2}{3}$ when 2 is subtracted from the numerator and 1 is added to the denominator, but equals $\frac{3}{2}$ when 2 is added to the numerator and 3 is subtracted from the denominator?

Let $x =$ the numerator,

and $y =$ the denominator.

Then $\frac{x}{y} =$ the fraction.

By the conditions, $$\frac{x-2}{y+1} = \frac{2}{3}, \tag{1}$$

$$\frac{x+2}{y-3} = \frac{3}{2}. \tag{2}$$

Clearing (1) and (2) of fractions, we have

$$3x - 6 = 2y + 2,$$

or $$3x - 2y = 8; \tag{3}$$

and $$2x + 4 = 3y - 9,$$

or $$2x - 3y = -13. \tag{4}$$

Multiplying (3) by 3, $$9x - 6y = 24. \tag{5}$$

Multiplying (4) by 2, $$4x - 6y = -26. \tag{6}$$

Subtracting (6) from (5), $$5x = 50.$$

$$\therefore\ x = 10.$$

Substituting this value in (3),

$$30 - 2y = 8.$$

$$\therefore\ -2y = -22,$$

and $$y = 11.$$

Hence the fraction is $\frac{10}{11}$.

3. A man can row 8 miles downstream in 40 minutes, and 14 miles upstream in 1 hour and 45 minutes. What is the rate of the current, and the rate of rowing in still water?

Let $x =$ the number of miles an hour rowed in still water,

and $y =$ the hourly rate in miles of the current.

Then $x + y =$ the number of miles an hour rowed downstream,

and $x - y =$ the number of miles an hour rowed upstream.

Since the number of miles rowed, divided by the hourly rate in miles, is equal to the number of hours occupied on each trip, we have

$$\frac{8}{x+y}=\frac{2}{3}, \tag{1}$$

and

$$\frac{14}{x-y}=\frac{7}{4}. \tag{2}$$

Solving, $x=10$, and $y=2$.

Hence the rate of the current is 2 miles an hour, and the rate of rowing in still water 10 miles an hour.

4. The sum of two numbers is 200, and their difference is 50. What are the numbers?

5. The sum of two numbers is $a+1$; their difference is $b-1$. What are the numbers?

6. If A gives B \$100, B will have twice as much money as A; but if B gives A \$100, A will have 3 times as much money as B. What sum of money has each?

7. With \$45 a man can buy either 20 yards of silk and 30 yards of flannel or 60 yards of flannel and 10 yards of silk. What is the price of silk?

8. When a certain number expressed by two digits is divided by the sum of its digits, the quotient is 3; and when the ones' digit is diminished by 3 times the tens' digit, the remainder is 1. Find the number.

9. A boy is given 66 cents to buy 2 dozen oranges. He pays 3 cents each for one kind and 2 cents each for the other. How many of the better kind did he buy?

10. The sum of the two digits of a certain number is 5. If the order of the digits is reversed, the number will be decreased 90%. Find the number.

11. A sum of money, at simple interest, amounted to \$3360 in 4 years, and to \$3450 in 5 years. Find the principal, and the rate of interest.

12. A man sold ducks at 50 cents each and chickens at 35 cents each. If the proceeds from the sale of 100 fowls amounted to $39.50, how many ducks and how many chickens did he sell?

13. Eight years ago A was 5 times as old as B, and in 2 years he will be 3 times as old. What are their ages?

14. The sum of two numbers is 16, and 7 times the quotient of the larger divided by the smaller is 9. What are the numbers?

15. What is the fraction which equals $\frac{2}{3}$ when 2 is added to the numerator, but equals $\frac{4}{13}$ when 4 is added to the denominator?

16. Tom said to Dick, "Give me 5 of your rabbits, and I shall have 3 times as many as you will have left." Dick replied, "No, but give me 2 of yours, and I shall have 5 times as many as you will have left." How many rabbits did each boy have?

17. A man and a boy can do a certain piece of work in 15 days. Seven men and 9 boys can do the same work in 2 days. How long would it take one man to do it?

18. In target practice, two marksmen, A and B, fire each 30 shots. If B makes twice as many hits as A, and A makes 3 times as many misses as B, find the number of hits made by each.

19. A man bought a house for $2100, paying for it with 135 bills, some of which were ten-dollar bills, some twenty-dollar bills. How many twenties were there?

20. If the numerator of a certain fraction is increased by 1, the value of the fraction becomes 1; but if the numerator is diminished by 1, the value of the fraction becomes $\frac{3}{4}$. What is the fraction?

21. A man sells at one time 4 sheep and 12 calves for $92, and at another time, at the same rate, 10 sheep and 3 calves for $68. What was the selling price of a sheep and of a calf?

22. What fraction is that whose numerator being doubled and its denominator increased by 7 the value becomes $\frac{2}{3}$, but the denominator being doubled and the numerator increased by 2 the value becomes $\frac{3}{5}$?

23. If a certain rectangle were 7 meters longer and 1 meter wider, it would contain 14 square meters more area, but if it were 1 meter shorter and $\frac{1}{2}$ meter wider, its total area would be unchanged. Find the dimensions.

24. Find two numbers in the ratio of 3 to 4 such that, if the first is increased by 7 and the second doubled, these numbers will be in the ratio of 2 to 3.

25. A boy spent \$3 for oranges, some at 30 cents a dozen and some at 40 cents a dozen. He sold them at 3 cents apiece, and gained 24 cents. How many dozen of each kind did he buy?

26. At the time of their marriage a husband's age was $\frac{4}{3}$ of his wife's age; 12 years afterward his age was $\frac{6}{5}$ of his wife's. How old was each at the time of marriage?

27. In an election 1490 men voted for two candidates, and the candidate elected had a majority of 246 votes. How many votes did each receive?

28. Find two numbers such that, if the first is increased by a, it will be m times the second, and if the second is increased by b, it will be n times the first.

29. A man has twenty coins, of which some are half dollars and the rest are nickels. If he should change the halves for nickels, and the nickels for cents, he would have 160 coins. How much money has he?

30. A and B engage in play. In the first game A wins as much as he had and \$4 more, and finds he has twice as much as B. In the second game B wins half as much as he had at first and \$1 more, when he has 3 times as much as A. What sum had each at first?

31. The income from a certain farm exceeds the expenses by $2000. If the income were diminished 20%, and the expenses diminished 25%, the excess of income over expenses would be $1800. Find the present income and expenses.

32. A crew that can row 10 kilometers an hour downstream finds that it takes twice as long to row up the stream as to row down. What is the rate of the current?

33. A pound of tea and 3 pounds of coffee cost $1.40; after an advance in price of 10% on coffee and 5% on tea they would cost $1.50. What is the price of tea and coffee?

34. A man invests a part of his capital at 3% and the rest at 4%, and thus secures an annual income of $370. If $1000 were taken from the second investment and added to the first, the incomes from the two investments would be the same. What is the amount of each investment?

35. The two digits of a number change places when 9 is subtracted from the number. The same result is obtained by adding 7 to one half of the number. What is the number?

36. An annual income of $120 is realized from a sum of money invested partly in $3\frac{1}{2}$% stock and partly in 4% stock. If the stock is sold when the first is at 108 and the second at 120, the sum received is $3672. How much stock of each kind was there?

---

37. In a certain meeting a motion was carried by a majority of 10 votes, but on a motion to reconsider, one fourth of those who had before voted in favor of the motion voted with the opponents, and the motion to reconsider was carried by a majority of 6 votes. How many voted each way on the first ballot?

38. A and B completed a certain piece of work in 30 days. If A had worked half as fast and B twice as fast, they would have finished the work in 24 days. How long would it take each alone to do the work?

39. A railway train, after traveling 1 hour, is detained 1 hour, after which it proceeds at $\frac{3}{5}$ of its former rate, and arrives 3 hours late. If the detention had occurred 50 miles farther on, the train would have arrived 1 hour and 20 minutes sooner than it did. Find the first rate of the train, and the distance traveled.

Let x = the number of miles an hour train ran at first,

and y = the number of miles traveled all together.

Then $y - x$ = the number of miles traveled after detention,

$\frac{y-x}{x}$ = the number of hours required to travel last distance at first rate,

and $\frac{5(y-x)}{3x}$ = the number of hours required to travel same distance at last rate.

Hence, by the conditions,

$$\frac{y-x}{x} + 2 = \frac{5(y-x)}{3x},$$

or
$$\frac{y-x}{x} - \frac{5(y-x)}{3x} = -2. \qquad (1)$$

Also,
$$\frac{y-x-50}{x} + (2 - 1\tfrac{1}{3}) = \frac{5(y-x-50)}{3x},$$

or simplifying,
$$\frac{y-x-50}{x} - \frac{5(y-x-50)}{3x} = -\frac{2}{3}. \qquad (2)$$

Subtracting (2) from (1), $\frac{50}{x} - \frac{250}{3x} = -\frac{4}{3}$,

whence $4x = 100.$

$\therefore x = 25.$

Substituting in (1), $y = 100.$

Hence the original rate of the train was 25 miles an hour, and the whole distance traveled was 100 miles.

40. A train, after running 5 hours, is detained 30 minutes, after which it proceeds at $\frac{3}{4}$ of its former rate, and arrives 3 hours late. If the detention had occurred 40 miles nearer the starting point, the train would have arrived $3\frac{1}{3}$ hours late. Find the original rate of the train, and the whole distance traveled.

41. A cistern is supplied by two pipes. If both pipes are opened for 15 minutes, they will fill $\frac{1}{6}$ of the cistern; but if the first pipe is opened for 12 minutes, and the second for 20 minutes, $\frac{1}{5}$ of the cistern will be filled. How long does it take each of the pipes alone to fill the cistern?

42. A train ran a certain distance at a uniform rate. If the speed had been 6 kilometers an hour more, the time occupied would have been 4 hours less; but if the speed had been 6 kilometers an hour less, the time occupied would have been 6 hours more. What was the distance traveled?

43. Two boys, A and B, run a mile, A giving B a start of 20 yards at the first trial, and beating him 30 seconds. At the second trial A gives B a start of 32 seconds, and beats him $9\frac{5}{11}$ yards. At what rates do A and B run?

44. Two trains start from two stations at the same time, each running at a uniform rate toward the other station. They meet in 12 hours, when one has gone 108 miles farther than the other. If they continue to travel at the same rate after meeting, they will finish their journey in 9 hours and 16 hours respectively. What is the rate of the trains, and the distance between the stations?

45. A man rows for $2\frac{1}{4}$ hours down a river which runs at a rate of 2 miles an hour. It takes him $3\frac{1}{2}$ hours to return to a point 5 miles below his starting place. Find his rate of rowing.

46. A cask contains 6 liters of wine and 8 liters of water. Another cask contains 4 liters of water and 12 liters of wine. How many liters must be drawn from each cask so that the mixture shall contain 5 liters of wine and 3 liters of water?

47. Two masons, A and B, are building a wall, which they could finish, working together, in 12 days. A works 3 days and B 2 days, when the wall is $\frac{1}{5}$ done. How long would it have taken each alone to build the wall?

48. If a certain number is divided by the sum of its two digits, the quotient is 6 and the remainder 3. If the digits

are interchanged and the number then expressed by them is divided by the sum of digits, the quotient is 4 and the remainder 9. What is the number?

49. A capitalist has \$36,750 to invest. He can buy 3% bonds at 75, and 5% bonds at 120. How much of his money must he invest in each kind of bonds in order to realize the same income from each investment?

50. After working 2 days on a job with B, A says to him, "I can finish this job alone in 10 days." B replies, "If we work together 1 more day, I can finish it alone in 5 days." If they are correct in their statements, how long would it take each alone to do the work?

51. A and B in a game of baseball are each at the bat 5 times. B makes twice as many hits as A, and A gets out $1\frac{1}{3}$ times as often as B. Find the number of hits and outs of each, assuming that each gets out when he does not make a hit.

52. Two trains, 200 feet and 152 feet long respectively, running at uniform rates on parallel tracks, pass each other in 3 seconds when running in opposite directions, and in 12 seconds when running in the same direction. Find the rate of each train.

53. In a certain meeting a motion was carried by a vote of 5 to 3. On reconsideration, 50 of the affirmative votes deserted to the negative. If 60 more had deserted, the motion would have been lost by a vote of 3 to 4. What was the vote on reconsideration?

54. A composition of lead and zinc, which weighs 149 pounds, loses 18 pounds in water. How many pounds are there of each metal, if a quantity of $11\frac{1}{3}$ pounds of lead and one of $6\frac{3}{4}$ pounds of zinc each loses 1 pound in water?

55. A banker has two kinds of money. It takes a pieces of one kind or b pieces of the other to make a dollar. If c pieces are given for a dollar, how many of each will be used? How many, if $a = 10$, $b = 4$, and $c = 7$?

56. On a certain street railway two sizes of cars are used. What is the seating capacity of each, if 14 more persons can be seated in 3 large cars than in 4 small cars, and 2 more persons in 2 large cars than in 3 small cars?

57. A vessel sailed with the wind and tide 60 miles, and returned with the wind and against the tide. She returned to the starting point in 12 hours, the rates of sailing out and in being as 5 to 3. Required the time each way, and the tide's rate an hour.

58. In football a "touchdown" counts 5 points and a "goal" 1 point in a score. In a game between two teams, A and B, the A team kicks 5 goals, and the B team makes 3 touchdowns, and the A team's score is $2\frac{1}{3}$ times that of B. If the A team had made no goals, its score would have doubled B's. What is the score?

59. Two wheelmen race a mile, and A wins by 5 seconds. In the second trial B has a start of 56 yards and wins by 2 seconds. In how many seconds does each ride a mile, assuming that each rides at a uniform rate?

60. A and B run a race of 300 meters. When A has a start of 3 seconds he is beaten by 21 meters, but when he has a start of 10 meters he is beaten by $6\frac{2}{3}$ seconds. Find the rate of each a second.

LINEAR EQUATIONS WITH THREE OR MORE UNKNOWN NUMBERS

233. We may, in general, solve *three* linear equations containing *three* unknown numbers by eliminating one of the unknown numbers from any two of the equations (usually the first and the second), and the same one from the third and either of the other two (usually the first), thus obtaining two linear equations containing only two unknown numbers. These may be solved by the methods already given.

When there are more than three unknown numbers, the process of elimination is similar.

1. Solve the system of equations

$$\begin{cases} 3x - 5y + 4z = 5. & (1) \\ 7x + 2y - 3z = 2. & (2) \\ 4x + 3y - z = 7. & (3) \end{cases}$$

First eliminate z from (1) and (2).

Multiplying (1) by 3, $9x - 15y + 12z = 15.$ (4)

Multiplying (2) by 4, $28x + 8y - 12z = 8.$ (5)

Adding (4) and (5), $37x - 7y = 23.$ (6)

Next eliminate z from (1) and (3).

Multiplying (3) by 4, $16x + 12y - 4z = 28.$ (7)

Adding (1) and (7), $19x + 7y = 33.$ (8)

Now eliminating y by adding (6) and (8), we have

$$56x = 56.$$

$$\therefore x = 1.$$

Substituting this value in (6), $37 - 7y = 23.$

$$\therefore y = 2.$$

Substituting 1 for x and 2 for y in (3),

$$4 + 6 - z = 7.$$

$$\therefore z = 3.$$

CHECK. Substituting in (1) and (2), we have

$$3 - 10 + 12 = 5, \text{ and } 7 + 4 - 9 = 2.$$

2. Solve the system of equations

$$\begin{cases} x + y = 2a. & (1) \\ y + z = 2b. & (2) \\ z + x = 2c. & (3) \end{cases}$$

Adding (1), (2), and (3),

$$2x + 2y + 2z = 2a + 2b + 2c.$$

Dividing by 2, $x + y + z = a + b + c.$ (4)

Subtracting (1), (2), and (3) respectively from (4),

$$z = -a + b + c,$$

$$x = a - b + c,$$

$$y = a + b - c.$$

Solve the following systems of equations:

3. $x-2y+3z=2.$
$2x-3y+z=1.$
$3x-y+2z=9.$

4. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=9.$
$\frac{2}{x}-\frac{3}{y}+\frac{4}{z}=11.$
$\frac{5}{x}+\frac{2}{y}-\frac{3}{z}=4.$

5. $x+y+z=9.$
$x+2y+3z=14.$
$x+3y+6z=20.$

6. $x+y+z-w=10.$
$x+y-z+w=14.$
$x-y+z+w=18.$
$y-x+z+w=22.$

7. $2x-3y+z=-1.$
$3x+y-4z=-7.$
$5x-y+3z=12.$

8. $x+2y=20.$
$3z-x=2.$
$2y+z=10.$

9. $\frac{1}{x+y}=1.$
$\frac{2}{x+z}=1.$
$\frac{3}{y+z}=1.$

10. $3x+2y=28.$
$8y-3z=19.$
$6z-4x=18.$

11. $7x-3y=1.$
$11z-7w=1.$
$5z+6y=134.$
$2x+w=33.$

12. $\frac{x}{a}+\frac{y}{b}=1.$
$\frac{x}{a}+\frac{z}{c}=1.$
$\frac{y}{b}+\frac{z}{c}=1.$

13. $9x+4y-10z=11.$
$12y-6x-5z=-9.$
$15z+3x+8y=-16.$

14. $x+y=11.$
$x+z=10.$
$y+z=9.$

15. $x+2y-3z=6.$
$2x+4y-7z=9.$
$3x-y-5z=8.$

16. $\frac{1}{x}+\frac{1}{y}=1.$
$\frac{1}{x}+\frac{1}{z}=2.$
$\frac{1}{y}+\frac{1}{z}=1\frac{1}{2}.$

17. $x+2y+3z=14.$
$2x+y+2z=10.$
$3x+4y-3z=2.$

18. $\frac{x}{6}+\frac{y}{3}+\frac{z}{4}=36.$
$\frac{x}{9}+\frac{y}{15}+\frac{z}{20}=10.$
$\frac{x}{4}+\frac{y}{2}+\frac{z}{10}=43.$

19. $x+2y+3z=32.$
$y+2z+3x=40.$
$z+2x+3y=40.$

20. $\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{10}.$
$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=\frac{3}{10}.$
$\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=\frac{11}{30}.$

21. $3x-2y+z=10.$
$2x+3y+z=18.$
$x+y+z=5.$

22. $x-y+z=2.$
$x+y-z=0.$
$x+y+z=10.$

23. $\frac{x}{2}-\frac{y}{3}+\frac{z}{4}=3.$
$\frac{3x}{4}+\frac{y}{6}-\frac{z}{3}=0.$
$\frac{5x}{6}-\frac{2y}{3}+\frac{7z}{12}=6\frac{1}{3}.$

24. $\frac{1}{x}+\frac{1}{y}-\frac{1}{z}=a.$
$\frac{1}{x}-\frac{1}{y}+\frac{1}{z}=b.$
$-\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=c.$

25. $x+y+z=m.$
$x-y-z=n.$
$x+y-z=p.$

26. $x+y+2z=2(m+n).$
$x+z+2y=2(m+p).$
$y+z+2x=2(n+p).$

27. $u+v+w+x+y=3.$
$u+v+w+x+z=8.$
$u+v+w+y+z=4.$
$u+v+x+y+z=9.$
$u+w+x+y+z=6.$
$v+w+x+y+z=5$

PROBLEMS

234. 1. A cistern has three pipes, A, B, and C. A and B will fill the cistern in 35 minutes, A and C in 42 minutes, and

B and C in 70 minutes. How long will it take each separately to fill it?

Let x, y, z = the number of minutes for A, B, and C respectively to fill it separately.

Then, $\frac{1}{x}, \frac{1}{y}, \frac{1}{z}$ = the parts A, B, C can fill in 1 minute respectively.

And $\frac{1}{x}+\frac{1}{y}$ = the part A and B can fill in 1 minute,

$\frac{1}{x}+\frac{1}{z}$ = the part A and C can fill in 1 minute,

and $\frac{1}{y}+\frac{1}{z}$ = the part B and C can fill in 1 minute.

But A and B can fill $\frac{1}{35}$ of it in 1 minute, A and C $\frac{1}{42}$ of it, and B and C $\frac{1}{70}$ of it.

Therefore, by the conditions,

$$\frac{1}{x}+\frac{1}{y}=\frac{1}{35}, \tag{1}$$

$$\frac{1}{x}+\frac{1}{z}=\frac{1}{42}, \tag{2}$$

$$\frac{1}{y}+\frac{1}{z}=\frac{1}{70}. \tag{3}$$

Adding (1), (2), and (3), $\frac{2}{x}+\frac{2}{y}+\frac{2}{z}=\frac{14}{210}$.

$$\therefore \frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{7}{210}. \tag{4}$$

Subtracting (1) from (4), $\frac{1}{z}=\frac{1}{210}$.

$\therefore z = 210.$

Subtracting (2) from (4), $\frac{1}{y}=\frac{1}{105}$.

$\therefore y = 105.$

Subtracting (3) from (4), $\frac{1}{x}=\frac{2}{105}$.

$\therefore x = 52\frac{1}{2}.$

Therefore, A can fill it in $52\frac{1}{2}$ minutes, B in 105 minutes, and C in 210 minutes.

2. A certain number represented by three figures is such that the sum of the ones' and hundreds' digits equals the tens'

digit; the ones' digit exceeds the hundreds' digit by 4, and the sum of the tens' and hundreds' digits is 2 more than the ones' digit. What is the number?

3. There are three numbers such that the sum of $\frac{1}{2}$ of the first, $\frac{1}{3}$ of the second, and $\frac{1}{4}$ of the third is 23; the sum of $\frac{1}{3}$ of the first, $\frac{1}{4}$ of the second, and $\frac{1}{5}$ of the third is 17, and the sum of the numbers is 68. Find the numbers.

4. Three cities, A, B, C, situated at the vertices of a triangle, are connected by straight roads. The distance from A to C by way of B is 118 kilometers, from B to A by way of C is 74 kilometers, and from C to B by way of A is 92 kilometers. Find the lengths of the three straight roads.

5. A certain number is expressed by three digits whose sum is 10. The tens' digit is equal to the sum of the ones' and hundreds' digits, and if 99 is added to the number, its digits will be reversed. Find the number.

6. A person finds he can buy for \$31.10 either 10 bushels of wheat, 12 bushels of rye, and 9 bushels of oats; or 12 bushels of wheat, 6 bushels of rye, and 13 bushels of oats; or 16 bushels of wheat, 10 bushels of rye, and 2 bushels of oats. What is the cost of a bushel of each?

7. A loaned his money at 4%, B at 5%, and C at 6%. If A and B together received \$1592 interest, B and C together \$1766 interest, and A and C together \$1638 interest, what sum did each loan?

8. A cistern has three pipes, A, B, and C. If A and B run in while C runs out, it will be filled in $28\frac{4}{7}$ minutes; if B and C run in while A runs out, it will be filled in 40 minutes; and if A and C run in while B runs out, it will be filled in $66\frac{2}{3}$ minutes. In what time would each alone fill it?

9. A number expressed by three digits, whose sum is 18, has its digits reversed if we add to it 22 times the sum of its digits; and the number so formed is 72 less than double the original number. What is the number?

10. A and B can do a piece of work in 12 days, B and C can do it in 18 days, and A and C can do half the work in 12 days. In how many days can each alone do the work?

11. A certain number consists of three digits, whose sum is 15. If the first two digits are reversed, the number becomes 180 larger; but if the last two digits are reversed, the number becomes but 18 larger. Find the number.

12. A, B, and C each had a pocket full of marbles. After each had given each of the others $\frac{1}{5}$ of the marbles, in his pocket, they found that A had 28, B 30, and C 32. How many had each at first?

13. A gave to B and C as much as each of them already had; B then gave to A and C as much as each of them then had; and C then gave to A and B as much as each of them had, after which each had \$16. How much had each at first?

14. A man invests \$7620 in bonds; he buys 4% bonds at 90, 5% bonds at 105, and 6% bonds at 120. How much is invested in each kind if he obtains the same income from each investment?

15. Two fractions having the same denominator are such that their sum is $\frac{3}{2}$. If 1 is added to each numerator, the sum of the fractions is $1\frac{3}{4}$; but if 1 is subtracted from each numerator, their difference is $\frac{1}{4}$. Find the fractions.

16. A, B, C, on a hunting trip killed 72 birds. In order to share them equally A gives to B and C as many as they already have; next, B gives to A and C as many as they each had after the first division; and then C gives to A and B as many as they each had after the second division, when each had the same number. How many had each at first?

17. Of three bars of metal, the first contains 2 kg. of silver, 3 kg. of copper, and 4 kg. of tin; the second contains 3 kg. of silver, 4 kg. of copper, and 5 kg. of tin; the third contains 4 kg. of silver, 3 kg. of copper, and 5 kg. of tin. How many kilograms is it necessary to take from each of these bars to form a bar which shall contain 9 kg. of silver, 10 kg. of copper, and 14 kg. of tin?

INDETERMINATE EQUATIONS

235. We have seen (Art. 218) that when a single linear equation contains two unknown numbers, an unlimited number of values of the unknown numbers may be found which satisfy the equation.

Such an equation is called an **Indeterminate Equation.**

236. Any required number of the roots of an indeterminate equation can usually be found by direct inspection.

For example, consider the equation

$$x+y=5,$$

which may be written $y=5-x.$

Then for $x=0,\ y=5$; for $x=1,\ y=4$; for $x=2,\ y=3$; for $x=3,\ y=2$; for $x=5,\ y=0$; for $x=6,\ y=-1$; and so on. It is thus seen that the value of either unknown number *depends upon that of the other.*

237. In general, if the number of independent equations is less than that of the unknown numbers, the solution is indeterminate. For instance, in the case of *two* equations, each containing *three* unknown numbers, we have, after eliminating either of the unknown numbers, a single equation containing only the other two unknown numbers.

Thus, from the equations

$$\begin{cases} 3x+2y+z=21, \\ 4x+\ \ y+z=13, \end{cases}$$

there arise the equations,

by eliminating z, $\quad -x+y=8$;

or eliminating x, $\quad 5y+z=45$;

or eliminating y, $\quad 5x+z=5$;

all of which are indeterminate.

238. Although the number of solutions of an indeterminate equation is in general unlimited, restrictions may be imposed upon the nature of the solution, as by requiring that the values of the unknown numbers be all integral, or integral and positive, which will render the solution indeterminate in a less general sense.

For practical purposes the values of the unknown numbers are usually limited to *positive integers*, and in this connection we shall consider only indeterminate linear equations in which such values are required.

239. The usual methods of finding the positive integral roots of indeterminate equations are illustrated in the following solutions:

1. Find the positive integral roots of $3x + 5y = 20$.

Since the equation may be written

$$5y = 20 - 3x,$$

it is seen that x cannot be greater than 6, and integral, for if $x > 6$, y is negative.

If $x = 6, 5, 4, 3, 2, 1$, then $y =$ a fraction in each case (which is readily seen by inspection), except for the value $x = 5$, when $y = 1$.

$\therefore$ $x = 5$, $y = 1$ are the only positive integral roots.

2. Find the positive integral roots of $3x + 4y = 34$.

Solving for x, because it has the smaller coefficient, we have

$$x = \frac{34 - 4y}{3} = 11 + \frac{1}{3} - y - \frac{y}{3} = 11 - y + \frac{1 - y}{3}.$$

$$\therefore\ x + y - 11 = \frac{1 - y}{3}. \qquad (1)$$

Since x and y are to be integers, $x + y - 11$ will be integral, and hence $\frac{1 - y}{3}$ will be integral, which is the case when y has such a value that $1 - y$ is divisible by 3.

Let $\frac{1 - y}{3} \equiv m$, an integer.

Then $y = 1 - 3m.$ (2)

Substituting this value in (1), $x = 10 + 4m.$ (3)

To make y and x positive integers, it is evident from (2) that

$$m = \left[\begin{matrix} 0, & -1, & -2, \\ 0, & -1, & -2. \end{matrix}\right] - 3 \cdots$$

From (3), $m = \cdots + 2, + 1,$

Therefore m has either of the values $0, -1, -2$.

If $m = 0$, $x = 10$ and $y = 1$;

$= -1$, $= 6$ $= 4$;

$= -2$, $= 2$ $= 7$.

3. Find the positive integral roots of $5x + 8y = 29$.

Here $$x = 5 - y + \frac{4 - 3y}{5},$$

and $$x + y - 5 = \frac{4 - 3y}{5}.$$

Let $$\frac{4 - 3y}{5} \equiv m, \text{ an integer.}$$

Then $$y = \frac{4 - 5m}{3}, \text{ which is not an easy form}$$

from which to find the integral values of y, and it remains to seek a form which will more readily yield these values. Since if $\frac{4 - 3y}{5}$ is an integer, any multiple of it is also an integer, we may reach the desired result by multiplying the numerator by 2 (which is the least number which will make the coefficient of y one more than a multiple of the denominator).

We then have $$\frac{8 - 6y}{5}, \text{ or } 1 - y + \frac{3 - y}{5}.$$

Now, let $$\frac{3 - y}{5} \equiv m, \text{ an integer.}$$

Then $$y = 3 - 5m. \tag{1}$$

Substituting in the original equation,

$$x = 1 + 8m. \tag{2}$$

From (1), $m = \left[\begin{matrix} 0, \\ 0. \end{matrix}\right] - 1, -2, \cdots$

From (2), $m = \cdots + 2, + 1,$

$\therefore$ $m = 0$ is the only value which will satisfy conditions (1) and (2).

When $m = 0$, $x = 1$, and $y = 3$.

4. Find the positive integral roots of the equations

$$\begin{cases} 2x + 3y + z = 15. & (1) \\ 3x + y - z = 8. & (2) \end{cases}$$

Eliminating z, $\quad 5x + 4y = 23.$ (3)

Solving (3) by the method of the preceding example, or by the simpler considerations of example 1, we obtain $x = 3$, $y = 2$.

Substituting these values in (1), $z = 3$.

Find the positive integral roots of the following:

5. $3x + 5y = 35.$
6. $2x + 3y = 25.$
7. $5x + 2y = 17.$
8. $3x + 4y = 22.$
9. $x + y + z = 6.$
   $3x + 2y - z = 12.$
10. $2x + 11y = 49.$
11. $3x + 7y = 58.$
12. $7x + 9y = 100.$
13. $5x + 9y = 37.$
14. $x + y + z = 20.$
    $5x + 4y + \frac{3}{2}z = 67.$

Find the least integral values of x and y in the following:

15. $5x - 3y = 11.$
16. $5x - 8y = 7.$
17. $10x - 3y = 11.$
18. $4x - 13y = 10.$
19. $9x - 5y = 21.$
20. $8x - 21y = 33.$

Solve the following equations in positive integers:

21. $3x + 4y = 39.$
22. $10x + 5y + z = 150.$
    $x + y - z = 0.$
23. $3x + 11y = 31.$
24. $2x + 3y - 5z = -8.$
    $5x - y + 4z = 21.$

25. A number is expressed by two digits. If it is divided by the number expressed by the digits reversed, the quotient is 2, and the remainder 2. What is the number?

26. Find two numbers which, when multiplied respectively by 12 and 5, have 75 for the sum of their products.

27. In how many ways can 22 pounds be weighed by using 3-pound and 4-pound weights?

28. In how many ways can a debt of \$27 be paid with two-dollar bills and five-dollar bills?

29. A man spent \$206 in buying calves at \$10 apiece and hogs at \$7 apiece. How many of each did he buy?

30. In how many ways may 110 be divided into two parts, one of which shall be a multiple of 7, the other a multiple of 16?

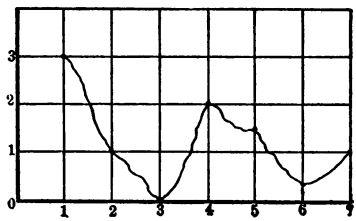
31. In how many different ways can a debt of $5.10 be paid with half dollars, quarters, and dimes, the whole number of coins used being 20?

32. What is the least number which, when divided by 3, leaves a remainder of 2, and divided by 5, leaves a remainder of 3?

GRAPHS OF LINEAR EQUATIONS

240. In practical affairs and in science much use is made of diagrams or graphic forms to exhibit at a glance the relations among several measurements or to bring into view for comparison the numerical results found by investigation.

For example, if the number of inches of snowfall on the seven days of a certain week are 3, 1, 0, 2, 1.5, 0.5, 1, we may represent these facts graphically as in the annexed diagram, in which the numbers denoting the depth of the fall on each day are measured off above the base line at the point indicating the corresponding day. The curved line drawn through the vertical points of divisions shows in an expressive way the comparative depth of the fall on the different days of the week.



241. In the same way the graphic method can be used to great advantage in representing the relations between the values of the unknown numbers in simple equations, and in exhibiting the solutions of the equations themselves, as will appear from what follows.

242. If we draw on paper a pair of lines XX', YY', intersecting at right angles at O, as in Fig. 1, we can pass from O to any point P, by first going horizontally a certain number of squares, and then vertically a certain number of squares. If the distance passed over horizontally is 2 inches, and that passed over vertically is 3 inches, it is seen that one pair of

numbers, 2, 3, is sufficient to locate the point P with reference to the lines XX' and YY'.

243. The above method can be applied to represent at the same time by one point a value of x and also a value of y, by measuring values of x on the horizontal line and values of y on the vertical line.

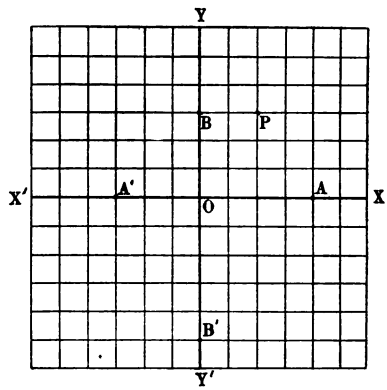


Fig. 1

244. In the figure the two lines XX', YY', intersecting at right angles at O, are called **Axes**; the line XX' being the *x-axis* and the line YY' the *y-axis*.

245. Any distance on OX from OY is the x-distance, or *abscissa*, as OA or OA', Fig. 1; any distance on OY from OX is the y-distance, or *ordinate*, as OB or OB'. The abscissa and ordinate together are called the **Coördinates** of any point, as P, the abscissa being named first.

In Fig. 1 the point P is 2 units to the *right* of OY, hence its abscissa is $+2$; it is 3 units *above* OX, hence its ordinate is $+3$. Its coördinates therefore are $+2$ and $+3$.

246. In the representation of points abscissas to the right of O are called positive, those to the left negative. Ordinates above XX' are called positive, those below negative.

Thus, in Fig. 2, the coördinates of P are 2 and 3, usually written in the form $(2, 3)$; those of P_1 are $(-2, 3)$; those of P_2 are $(-2, -3)$; and those of P_3 are $(2, -3)$. Evidently the abscissa of any

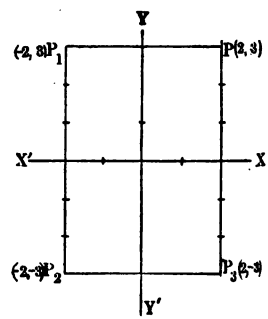


Fig. 2

point on YY' is 0, and the ordinate of any point on XX' is also 0. The coördinates of O are (0, 0). How do the pairs of coördinates of P, P_1, P_2, and P_3 differ?

247. It follows from the preceding that a point in any given plane is fixed when its coördinates are known, the point being designated by its coördinates.

Thus, the point P is designated by (2, 3), the point P_1 by $(-2, 3)$, etc. If the coördinates are unknown, they are designated by x and y, written (x, y).

248. To plot the point P whose coördinates are (2, 3), we measure 2 units on the x-axis to the right of O, and on the vertical line (parallel to the y-axis) through the point thus reached we measure 3 units up from XX'. Similarly for other points.

Using a convenient unit of measure, plot the coördinates of the following points:

1. (3, 0), (0, 3).
2. (3, 5), $(-3, -5)$.
3. (3.5, 6), $(-2, 3)$.
4. $(-3, 0)$, (0, 2).
5. $(5, -4)$, $(-1, -7)$.
6. (3, 4), $(-3, -1)$.
7. $(-4, 6)$, (4, 0).
8. (5, 0), $(0, -2)$.

249. Graph of an Equation. In the indeterminate equation $y = 2x - 3$, if $x = 3, 2, 1, 0, -1, -2, -3, \cdots$, then $y = 3, 1, -1, -3, -5, -7, -9$, respectively.

This relation is represented graphically in Fig. 3, in which the values of x are measured along the horizontal axis OX (to the right of O or to the left of

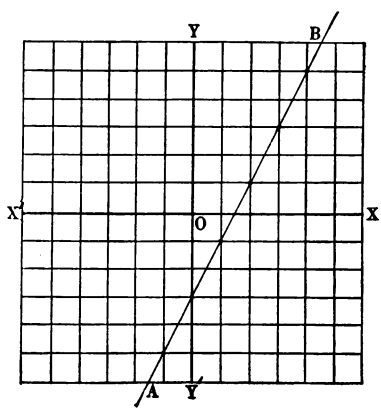


FIG. 3

O, according as the value is positive or negative) and the corresponding values of y are measured in the vertical direction (upward or downward, according as the value is positive or negative). It is seen that to each pair of values (roots) corresponds a point, and the points thus located all lie on the straight line representing the equation.

This line, as AB, is called the **Graph of the Equation.**

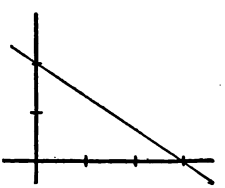
250. The graph of a simple equation containing two unknown numbers is *always a straight line.* For this reason it is called a *linear* equation, although the term has been extended to include all equations of the first degree.

251. Since a linear equation is always represented by a straight line, we need to know only two points on the line to determine the graph of the equation.

In general, the two points most easily determined are those at which the graph cuts the axis of coördinates. These points are found by first letting $x = 0$ and finding the corresponding value of y; then letting $y = 0$ and finding the corresponding value of x.

Thus, in the equation $2x + 3y = 6$,
if $x = 0$, $y = 2$;
if $y = 0$, $x = 3$.

Hence, to plot the graph, draw a line through (0, 2) and (3, 0) as in the figure.



252. From what precedes, it is evident that the coördinates of any point on the graph satisfy the equation. And since a single straight line has an indefinite number of points, so a single linear equation containing two unknown numbers has an indefinite number of roots.

Draw the graphs of the following equations:

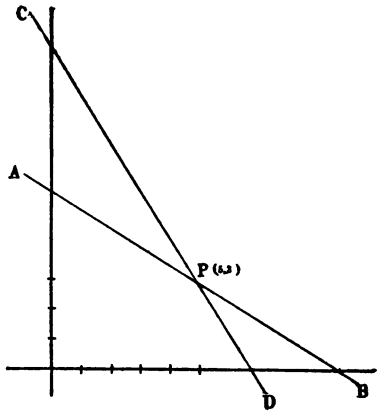
1. $x + y = 0$.
2. $3x + 4y = 12$.
3. $2x - 5y = 10$.
4. $y - 2x = 0$.
5. $2x + y = 1$.
6. $y = -4x + 6$.
7. $3x - 2y = 1$.
8. $4x - 6y = 1$.

253. Simultaneous Linear Equations.

If we draw the graphs of the two equations

$$x + 2y = 11,$$
$$2x + y = 13,$$

on the same diagram, as in the annexed figure, we find that the two straight lines representing the equations intersect at a point P whose coördinates are (5, 3), and that the values $x = 5$, $y = 3$ are the roots of the equations.



254. Since two straight lines have in general but one point in common, so two linear equations containing two unknown numbers have in general but one common pair of roots, the values of the roots being the same as the coördinates of the common point of the graphs.

Hence, it is seen that to obtain the solution of a system of equations containing two unknown numbers by means of graphs, it is merely necessary to plot the graphs of the two equations, and measure the coördinates of their intersection, attaching the proper signs to the values found.

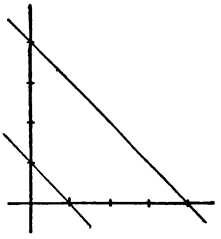
Draw the graphs of each of the following pairs of equations, and ascertain that they intersect in the point given by the algebraic solution:

1. $2x + y = 10.$
   $4x - 3y = 0.$
2. $x + y = -1.$
   $x + 2y = -4.$
3. $y = 3x - 2.$
   $y = -2x - 1.$
4. $x + 2y = 8.$
   $2x + y = 7.$
5. $2x - 3y = 7.$
   $3x - 4y = 13.$
6. $x + 9y = 13.$
   $3x + y = 14.$
7. $2x + 3y = 7.$
   $3x + 4y = 10.$
8. $3x - y = 4.$
   $x + 3y = -2.$
9. $5x + 4y = 22.$
   $3x + y = 9.$

10. Draw on the same axes the graphs of $y=x+3$, $y=x-3$, $y=-x-3$, $y=-x+3$, and find the number of square units of area inclosed by the four graphs.

255. If we attempt to solve the two equations

$$2x+2y=2,$$
$$2x+2y=8,$$



we find that we cannot eliminate either of the unknown numbers without eliminating the other. The graphs of the equations show at once the source of the difficulty. The lines representing the equations are *parallel*, and do not have a common point. This means that the equations are *inconsistent* (Art. 221), and hence do not have a common pair of roots.

256. Again, if we attempt to draw the graphs of the equations

$$x+2y=3,$$
$$3x+6y=9,$$

we find that they have an indefinite number of points common, every point on either graph being also a point on the other. Hence the graphs *coincide*. This is as it should be, for, as applied to the equations, it means that every solution of either is a solution of the other; hence that the two equations are *equivalent*, the second being derived from the first by multiplying both members of the latter by 3. The two equations reduce to a single *indeterminate equation*.

Draw the graphs of the following equations, and point out the systems that are inconsistent or indeterminate:

1. $2x+3y=8.$ $3x+2y=6.$
2. $x+y=1.$ $x-y=1.$
3. $x-2y=-1.$ $3x-6y=-3.$
4. $2x+2y=12.$ $3x+3y=18.$
5. $x+y=5.$ $3x+3y=16.$
6. $7x-9y=6.$ $x+2y=14.$

INVOLUTION AND EVOLUTION

INVOLUTION

257. A **Power** has already been defined (Art. 18) as the product of two or more *equal* factors.

By this definition the word "power" means positive integral power.

258. The operation of finding a power of a number or of an expression is called **Involution.**

259. Involution being merely a particular case of continued multiplication in which the factors are *equal*, several important principles that apply in involution have already been considered.

The following important laws of exponents are here summarized:

260. Laws of Exponents.

I. **The *n*th power of the *m*th power of an expression equals the *mn*th power of the expression.**

It has already been proved (Art. 114) that when m and n are positive integers,

$$(a^m)^n \equiv a^{mn}.$$

II. **The *m*th power of a product equals the product of the *m*th powers of its factors.**

It has also been proved (Art. 115) that when m is a positive integer,

$$(ab)^m \equiv a^m b^m.$$

III. **The *n*th power of a fraction equals the *n*th power of the numerator divided by the *n*th power of the denominator.**

In connection with the multiplication of fractions (Art. 186), it was proved that when n is a positive integer,

$$\left(\frac{a}{b}\right)^n \equiv \frac{a^n}{b^n}.$$

The proofs of these laws should be carefully reviewed by the student.

261. Laws of Signs.

Since $+a \times +a \equiv +a^2$, and $-a \times -a \equiv +a^2$, but $-a \times -a \times -a \equiv -a^3$, and so on, the following laws are evident (Art. 116):

I. **All powers of positive numbers are positive.**

II. **Even powers of negative numbers are positive, but odd powers are negative.**

262. Since $(a^2)^2 \equiv a^4$, and $(a^3)^2 \equiv a^6$, it is readily seen that in raising expressions to the 4th power we may conveniently square the expression, and then square that result; and in raising to the 6th power we may first cube, and then square the result (Art. 117).

Raise to the indicated power:

1. $(2a^2x^3y)^2$.
2. $(-2a^2b^3)^3$.
3. $(a+b)^4$.
4. $(a+b+1)^2$.
5. $(a^2+a+1)^2$.
6. $\left(\frac{x^2y^3}{a^3b^2}\right)^2$.
7. $\left(\frac{-3m^3x}{-5np^2}\right)^3$.
8. $\left(-\frac{3a^2b^3}{4pq^2}\right)^3$.
9. $(-3m^2n)^3$.
10. $(10ab^2c^3)^4$.
11. $(-2ab^2c^3)^5$.
12. $(2x+1)^4$.
13. $(xy^2z^3)^n$.
14. $\left(-\frac{3a^2b^3}{2c^4}\right)^4$.
15. $\left(\frac{3a^2b^3c^4}{5mn^2p^3}\right)^2$.
16. $\left(-\frac{2x^m}{3y^2}\right)^4$.
17. $(-\frac{1}{2}a^3x^m)^4$.
18. $(m-2n)^6$.
19. $(a-b+c)^2$.
20. $(a-2b+c)^3$.
21. $(a-b+x-y)^2$.
22. $\left(\frac{5a^2b^3}{6x^2y^3}\right)^5$.
23. $\left(\frac{a^2b^2}{xy}\right)^m$.
24. $\left(\frac{x^ny^n}{ab^2}\right)^n$.

263. By the Binomial Theorem.

It was shown (Art. 119) that, by the aid of this important theorem, binomials may be raised to any power without the trouble of multiplication. It was also seen that, by properly grouping the terms, expressions of three or more terms can frequently be treated as binomials.

264. From the inductive exercises there given it appears that, if the binomial $a+b$ is raised to the nth power, the result is expressed in the formula

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\times 3}a^{n-3}b^3 \cdots,$$

n being a positive integer.

In these results the following laws are obvious:

I. **The number of terms in the second member is 1 more than the binomial exponent.**

II. **The exponent of a in the first term, as well as that of b in the last term, is equal to the binomial exponent. The exponents of a decrease by 1 from term to term, while those of b increase by 1.**

III. **The first coefficient is 1, the second is equal to the binomial exponent; and the coefficient of any succeeding term is found by multiplying the coefficient of the preceding term by the exponent of a in that term, and dividing by 1 more than the exponent of b.**

The student should carefully review Art. 119.

1. Raise $(2x-3y)$ to the 3d power.

$$(2x-3y)^3 \equiv (2x)^3 + 3(2x)^2(-3y) + 3(2x)(-3y)^2 + (-3y)^3$$
$$\equiv 8x^3 - 36x^2y + 54xy^2 - 27y^3.$$

2. Square $x+y-z$.

$$(x+y-z)^2 \equiv [(x+y)-z]^2 \qquad \text{(Art. 119, Ex. 2)}$$
$$\equiv (x+y)^2 - 2(x+y)(z) + z^2$$
$$\equiv x^2 + 2xy + y^2 - 2xz - 2yz + z^2.$$

Explain how the trinomial is treated as a binomial in this solution.

Raise to the indicated powers:

3. $(x-2y)^2$.
4. $(1+x+x^2)^2$.
5. $(1+x)^4$.
6. $(2a+3)^3$.
7. $(2a^2-3b^2)^3$.
8. $(1-a+a^2)^3$.
9. $(a^3+b^3+c^3)^3$.
10. $(3ab^2-4c^3d^4)^3$.
11. $(4a-3b^3)^3$.
12. $\left(1-\frac{a^2}{2m}\right)^3$.
13. $\left(x^2+\frac{x}{c^2}\right)^3$.
14. $(m-n-2)^3$.
15. $(a+2b+3c)^3$.
16. $(3x+5y-7z)^2$.
17. $(x^2-3x-1)^3$.
18. $\left(\frac{1}{a}-\frac{1}{b}\right)^4$.
19. $\left(ab-\frac{c}{2}\right)^4$.
20. $(a+b+c+d)^3$.
21. $(mx-ny)^5$.
22. $(2m+3n)^5$.
23. $(1-y)^8$.
24. $(x+1)^n$.

EVOLUTION

265. A **Root** has already been defined (Art. 20) as one of the equal factors of a number. The *square root* is one of the two equal factors; the *cube root* is one of the three equal factors; and so on.

266. The operation of finding a root of a number or of an expression is called **Evolution**. It is the inverse of Involution.

Thus, by involution we find that $2 \times 2 \times 2$, or $2^3 = 8$, $(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$; and by evolution we find one of the 3 equal factors of 8 or of $a^3 + 3a^2b + 3ab^2 + b^3$; that is, we find the cube root.

267. Just as involution is a particular case of multiplication, so evolution is merely a special case of factoring in which the factors are equal.

268. As previously stated (Art. 21), evolution is indicated by the symbol $\sqrt{\ }$, but we shall see in the study of the Theory of Exponents that both involution and evolution can also be indicated by exponents.

The number written over the symbol $\sqrt{\ }$ to indicate what root is to be taken is called the **Index** of the root. The index 2 is usually omitted; thus, the square root of a is indicated by $\sqrt{a}$, the cube root by $\sqrt[3]{a}$, and the nth root by $\sqrt[n]{a}$.

269. Law of Exponents.

It is shown in Art. 114 that when m and n are positive integers,

$$(a^m)^n \equiv a^{mn}.$$

Since evolution is the *inverse* of involution, it follows from this identity that

$$\sqrt[n]{(a^m)^n} \equiv a^m \equiv a^{\frac{mn}{n}}, \quad a^m \text{ being one value of } \sqrt[n]{a^{mn}}.$$

Thus, $$\sqrt[3]{a^{2\times 3}} \equiv \sqrt[3]{a^6} \equiv a^2.$$

It is shown in Art. 115 that

$$(ab)^n \equiv a^n b^n,$$

from which it follows that

$$\sqrt[n]{a^n b^n} \equiv \sqrt[n]{(ab)^n} \equiv ab.$$

But $$\sqrt[n]{a^n} \times \sqrt[n]{b^n} \equiv ab.$$

$$\therefore \sqrt[n]{a^n b^n} \equiv \sqrt[n]{a^n} \times \sqrt[n]{b^n} \equiv ab.$$

Thus, $$\sqrt[3]{a^3 b^3} \equiv \sqrt[3]{a^3} \times \sqrt[3]{b^3} \equiv ab.$$

What is the nth root of $a^n b^{2n}$, n being even? n being odd?

It is shown in Art. 186 that

$$\left(\frac{a}{b}\right)^n \equiv \frac{a^n}{b^n},$$

from which it follows that

$$\sqrt[n]{\frac{a^n}{b^n}} \equiv \sqrt[n]{\left(\frac{a}{b}\right)^n} \equiv \frac{a}{b}.$$

But $$\frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} \equiv \frac{a}{b}.$$

$$\therefore \sqrt[n]{\frac{a^n}{b^n}} \equiv \frac{\sqrt[n]{a^n}}{\sqrt[n]{b^n}} \equiv \frac{a}{b}.$$

Thus, $$\sqrt[3]{\frac{a^3}{b^3}} \equiv \frac{\sqrt[3]{a^3}}{\sqrt[3]{b^3}} \equiv \frac{a}{b}.$$

270. Laws of Signs.

It is evident from the laws of signs for involution that:

I. **An even root of a positive number is either positive or negative.**

Thus, $\sqrt{9} = \pm 3$, for $3 \times 3 = 9$, and $-3 \times -3 = 9$.

II. **An odd root of a number has the same sign as the number itself.**

Thus, $\sqrt[3]{27} = 3$; $\sqrt[3]{-27} = -3$.

III. **An even root of a negative number is neither a positive nor a negative number.** It can only be *indicated,* and is said to be *imaginary.*

Thus, $\sqrt{-1}$ is neither $+1$ nor -1. Even roots of negative numbers will be discussed later (Art. 327).

271. From what precedes it is evident that to extract the root of a monomial expression, we need merely to *divide the exponent of each letter by the index of the root, take the required root of the numerical coefficient, and prefix the proper sign.*

272. Roots by Factoring Method.

The root of a monomial is easily found by the inspection of equal factors. Similarly, the roots of polynomial expressions which are easily separated into equal factors, can often be found by inspection.

1. Extract the square root of $9a^2b^4$.

By factoring, $9a^2b^4 \equiv 3 \times 3 \times a \times a \times b \times b \times b \times b$.

$$\therefore \sqrt{9a^2b^4} \equiv \sqrt{(3 \times a \times b \times b)(3 \times a \times b \times b)} \equiv \pm 3 \times a \times b \times b$$
$$\equiv \pm 3ab^2.$$

Why is $3 \cdot 3 \cdot a \cdot a \cdot b \cdot b \cdot b \cdot b \equiv (3 \cdot a \cdot b \cdot b)(3 \cdot a \cdot b \cdot b)$?

2. Find the square root of $36x^6 + 24x^3y + 4y^2$.

Since $[\pm(a+b)]^2 \equiv a^2 + 2ab + b^2$ is the type of the perfect square of any binomial, it follows that the roots of expressions that can be arranged in this form can be found by inspection.

Here we see that

$$36x^6 + 24x^3y + 4y^2 \equiv (6x^3)^2 + 2(6x^3)(2y) + (2y)^2,$$

which is evidently the square of $\pm(6x^3 + 2y)$.

$\therefore \pm(6x^3 + 2y)$ is the required root.

3. Find the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

Since $(a \pm b)^3 \equiv a^3 \pm 3a^2b + 3ab^2 \pm b^3$ (Art. 118, IV, V), it follows that the cube root of expressions that can be arranged in the form of this type can be found by inspection.

Here we see that

$$8x^3 - 36x^2y + 54xy^2 - 27y^3 \equiv (2x)^3 - 3(2x)^2(3y) + 3(2x)(3y)^2 - (3y)^3$$
$$\equiv (2x - 3y)^3.$$

$\therefore 2x - 3y$ is the root required.

Find the required root by factoring:

4. $\sqrt{4a^2x^4}$.

5. $\sqrt[3]{8a^6}$.

6. $\sqrt[4]{16x^4y^8}$.

7. $\sqrt[3]{-27m^6}$.

8. $\sqrt[5]{-x^{10}y^{15}}$.

9. $\sqrt[3]{216x^3y^6z^9}$.

10. $\sqrt[4]{\dfrac{a^8}{81b^{12}}}$.

11. $\sqrt[3]{\dfrac{27x^3y^6}{64a^6b^9}}$.

12. $\sqrt[5]{-32x^5y^{10}}$.

13. $\sqrt[6]{64x^{12}y^{24}}$.

14. $\sqrt[5]{243a^{10}b^{15}}$.

15. $\sqrt[n]{a^{mn}x^{2n}}$.

16. $\sqrt[3]{27(x-2y)^6}$.

17. $\sqrt[3]{1728a^{3m}b^{6n}}$.

18. $\sqrt[7]{\dfrac{a^7}{128b^{14}}}$.

19. $\sqrt{\dfrac{16a^2b^4}{25a^8}}$.

20. $\sqrt[7]{-m^{35}n^{21}p^{14}q^7}$.

21. $\sqrt[3]{\dfrac{27x^3y^6}{-125a^9}}$.

22. $\sqrt[n]{\dfrac{m^{2n}x^{3n}}{n^ny^{4n}}}$.

23. $\sqrt{9a^2 + 30a + 25}$.

24. $\sqrt{m^4 + 4m^2n + 4n^2}$.

25. $\sqrt{36x^6 + 24x^3y + 4y^2}$.

26. $\sqrt{9x^2 - 2xy + \frac{1}{9}y^2}$.

27. $\sqrt{49x^8 + 14x^4 + 1}$.

28. $\sqrt[3]{x^3 - 15x^2 + 75x - 125}$.

273. Square Root of Polynomials.

Since $[\pm(a + b)]^2 \equiv a^2 + 2ab + b^2$, the square root of $a^2 + 2ab + b^2$ is known to be $\pm(a + b)$. Hence to discover a method for finding the square root of any polynomial it is necessary to reverse the process of squaring $a + b$. From an inspection of the above identity it is seen that:

(1) The first term of the square contains the square of the first term of the root. By reversing the process it follows that the first term of the root is the square root of the first term of the expression.

(2) The second term of the square contains twice the product of the two terms of the root. It follows that if the second term of the square, $2\,ab$, is divided by twice the first term of the root, $2\,a$, the quotient will be the second term of the root.

(3) If to $2\,a$ we add b, when it has been found, and then multiply the result by b, the product is equal to the remaining terms after a^2 has been subtracted. This is true, since the remainder $2\,ab + b^2 \equiv (2\,a + b)b$.

The relation between the terms of the binomial and the terms of its square will be more readily discovered from the following arrangement of the work:

Power	$= a^2 +$	$(2\,a + b)b$	$a + b =$ Root
	a^2		
Trial Divisor	$= 2\,a$	$(2\,a + b)b$	
	$+ b$		
Complete Divisor	$= 2\,a + b$	$(2\,a + b)b$	

Since a^2 is the square of the first term of the root, the first term, a, is obviously found by taking the square root of a^2, the first term of the power. Subtracting a^2, we have the remainder, $2\,ab + b^2$, or $(2\,a + b)b$.

The second term of the root, which is known to be b, may now be found by dividing the first term of the remainder, $2\,ab$, by $2\,a$, that is, by *double the root already found.* This divisor is called a *trial* divisor.

Now, since $2\,ab + b^2 \equiv (2\,a + b)b$, the complete divisor *is found by adding to the trial divisor the second term of the root.*

Multiplying the complete divisor by the second term of the root, and subtracting the product from the first remainder, $2\,ab + b^2$, we have the remainder 0. Hence, the root is $\pm(a + b)$.

274. Since $[\pm(a + b)]^2 \equiv a^2 + 2\,ab + b^2 \equiv a^2 + (2\,a + b)b$ represents the perfect square of any number, it may be used as a type, or formula, for finding the square root of any expression, provided we always let a equal the *part of the root already found* at any stage of the process, and b *the next term to be found.*

1. Find the square root of $9x^4 - 24x^2y^3 + 16y^6$.

$$\begin{array}{rl|l} & 9x^4 - 24x^2y^3 + 16y^6 & 3x^2 - 4y^3 \\ a^2 = (3x^2)^2 = 9x^4 & & \\ \hline 2a = 2(3x^2) = 6x^2 & -24x^2y^3 + 16y^6 & = (2a + b)b \\ 2a + b = 6x^2 - 4y^3 & -24x^2y^3 + 16y^6 & \end{array}$$

Since $9x^4$ is the square of the first term of the root, that term is $3x^2$.

The trial divisor is $2(3x^2)$, or $6x^2$. Therefore, the second term of the root is $-24x^2y^3 \div 6x^2$, or $-4y^3$.

$\therefore \pm(3x^2 - 4y^3)$ is the required root.

2. Find the square root of $9x^4 - 12x^3 + 10x^2 - 4x + 1$.

$$\begin{array}{rl|l} & 9x^4 - 12x^3 + 10x^2 - 4x + 1 & 3x^2 - 2x + 1 \\ & 9x^4 & \\ \hline 6x^2 & -12x^3 + 10x^2 & \\ 6x^2 - 2x & -12x^3 + 4x^2 & \\ \hline 6x^2 - 4x & & 6x^2 - 4x + 1 \\ 6x^2 - 4x + 1 & & 6x^2 - 4x + 1 \end{array}$$

After the first two terms of the root have been found as in the preceding solution, the next term is obtained by treating the part of the root already found, $3x^2 - 2x$, as the first term in finding the next term of the root. Hence the new trial divisor is $6x^2 - 4x$, and the third term of the root is 1.

To form the complete divisor, we annex 1 to the trial divisor already found, thus obtaining $6x^2 - 4x + 1$. We then multiply this complete divisor by the last term of the root.

$\therefore$ the required root of the given expression is $\pm(3x^2 - 2x + 1)$.

How should the polynomial be arranged so that the square root of the first term shall be the first term of the root?

Find the square root of the following expressions:

3. $x^4 - 4x^3 + 2x^2 + 4x + 1$.

4. $x^4 - 2x^3y + 3x^2y^2 - 2xy^3 + y^4$.

5. $16x^4 - 24x^3b + 25x^2b^2 - 12xb^3 + 4b^4$.

6. $a^2 - 2ab^2 + b^4 + 4ac - 4b^2c + 4c^2$.

7. $4x^4 + 12x^3 - 3x^2 - 18x + 9$.

8. $4x^4 - 4x^3 + 5x^2 - 2x + 1$.

9. $1 - 4m + 10m^2 - 12m^3 + 9m^4$.

10. $a^4 + 4a^3b + 10a^2b^2 + 12ab^3 + 9b^4$.

11. $81 - 18a + a^2 + 18b - 2ab + b^2$.

12. $x^4 + 4x^3 + 4x^2 + 2x + 4 + \frac{1}{x^2}$.

13. $x^2 + 2x - 1 - \frac{2}{x} + \frac{1}{x^2}$.

14. $a^2 + b^2 + c^2 + 2ab + 2ac + 2bc$.

15. $4x^6 + 12x^5 + x^4 - 24x^3 - 14x^2 + 12x + 9$.

16. $16a^6 - 24a^5 + 25a^4 - 20a^3 + 10a^2 - 4a + 1$.

17. $9(x^2 - 1)^2 - 12(x^2 - 1)x + 4x^2$.

18. $x^4 + x^3 + \frac{5}{4}x^2 + \frac{1}{2}x + \frac{1}{4}$.

19. $9a^2 + 25b^2 + 16c^2 + 30ab - 24ac - 40bc$.

20. $\frac{m^2}{4} - mx + \frac{3m}{2} + x^2 - 3x + \frac{9}{4}$.

21. $-6y^2z^2 + 9z^4 + y^4 - 12z^2x^2 + 4x^4 + 4x^2y^2$.

22. $n^8 + 2n^7 + n^6 - 4n^5 - 12n^4 - 8n^3 + 4n^2 + 16n + 16$.

275. Square Roots of Arithmetical Numbers.

The method of finding the square root of arithmetical numbers is derived from the identity

$$(t + o)^2 \equiv t^2 + 2to + o^2 \equiv t^2 + (2t + o)o,$$

in which t denotes the number of units expressed by the tens' digit and o the number of units expressed by the ones' digit. The method is similar to that for finding the square root of algebraic expressions, and will be understood from the following illustrative solutions:

Find the square root of 1369.

(*a*)

	$t + o$
13 69	$30 + 7 = 37$
$9\ 00 = t^2$	
$2t = 2 \times 30 = 60$	$4\ 69 = (2t + o)o$
$o =$ 7	
$2t + o$ $= 67$	$4\ 69$

(*b*)

13 69	37
9	
67	4 69
	4 69

As learned in arithmetic, the four figures of the number show that the root is expressed by *two* figures. In 1300 the greatest tens-square is 900, and its square root is 30, which is the tens of the root.

The square of the tens, t^2, is subtracted, and the remainder, 469, contains $2\,to + o^2$; that is, twice the tens × the ones + the square of the ones. But 469 cannot be separated into the sum of two terms, one of which is $2\,to$ and the other o^2, and hence o cannot be definitely determined by dividing $2\,to$ by $2\,t$, as in algebraic expressions. However, it is approximately determined by *doubling the partial factor* $30(=t)$, which is a part of $2\,t + o$ in $(2\,t + o)o$, for a *trial* divisor, since of the two factors – twice the tens × the ones – twice the tens, or 60, is so much the larger that it will probably give the value of o when so used. Using it thus, we find 7 to be the ones' figure of the root. Since $2\,to + o^2$ is equal to $(2\,t + o)o$, the seven ones are added to twice the tens, and the sum, 67, is multiplied by 7. In using the trial divisor, if the quotient is found to be too great, we must diminish it.

In practice the operation is conveniently performed as in (*b*), omitting unnecessary ciphers. The first group, 13, contains the square of the tens' number of the root. The greatest square in 13 is 9, and its square root is 3. The square of the tens is subtracted. Twice the 3 tens is 6 tens, and 6 tens is contained in the 46 tens of the remainder 7 times, giving the ones' figure of the root.

276. We may apply the method given above to numbers of more than two groups of figures by always regarding *the part of the root already found as so many tens* with respect to the next figure of the root.

What is the square root of 54,756?

(*a*)

	5 47 56 | 234
	4
2 × 20 = 40	1 47
40 + 3 = 43	1 29
2 × 230 = 460	18 56
460 + 4 = 464	18 56

(*b*)

5 47 56	234
4	
43	1 47
	1 29
464	18 56
	18 56

The first trial and complete divisors are obtained as they would be if the given number were 547; that is, $t = 20$ and $o = 3$. For the second divisors, $t = 230$ and $o = 4$.

277. From the preceding we may derive the following method of finding the square root of arithmetical numbers:

1. *Beginning at the ones, separate the number into groups of two figures each.*

2. *Find the greatest square in the left-hand group, and write its root for the first part of the required root.*

3. *Subtract the square of this root from the left-hand group, and to the remainder annex the next group for a dividend.*

4. *Divide this dividend by twice the root already found, considered as tens. The quotient (or the quotient diminished) will be the next figure of the root.*

5. *To the last trial divisor add the part of the root last found for a complete divisor. Multiply this complete divisor by the part of the root last found, subtract the product from the dividend, to the remainder annex the next group for a new dividend, and proceed as before until all of the groups have been thus annexed.*

(1) A decimal is separated into groups of two figures each by beginning at the decimal point, and its root is found precisely as the root of an integer is found.

(2) The exact square root cannot be found if a number contains an *odd* number of decimal places or if there is a remainder after the number of root figures found equals the number of groups. In such cases, however, we may find the root approximately by annexing ciphers and continuing the operation.

(3) The square root of a common fraction is found by extracting the square root of numerator and denominator separately, or by reducing it to a decimal and then finding the root.

Find the square root of the following:

1. 1225.	4. 15,625.	7. 5416.96.	10. $\frac{169}{196}$.
2. 1849.	5. 60,516.	8. 97.8121.	11. $\frac{324}{2916}$.
3. 9216.	6. 160,801.	9. 41.2164.	12. $3\frac{6}{25}$.

278. Cube Root of Polynomials.

Since $(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3$ (Art. 118, IV), the cube root of $a^3 + 3a^2b + 3ab^2 + b^3$ is known to be $a + b$. To discover a method for finding the cube root of a polynomial it is necessary to reverse the process of cubing $a + b$.

The relation between the terms of the binomial and the terms of its cube will more readily be discovered from the following arrangement of the work:

$$\begin{array}{lll} \text{Power} & = & a^3 + (3a^2 + 3ab + b^2)b \,|\, \underline{a + b = \text{Root.}} \\ & & a^3 \\ \text{Trial Divisor} & = 3a^2 & |(3a^2 + 3ab + b^2)b \\ & \quad + 3ab + b^2 & | \\ \text{Complete Divisor} & = \underline{3a^2 + 3ab + b^2} & |\underline{(3a^2 + 3ab + b^2)b} \end{array}$$

Since a^3 is the cube of the first term of the root, the first term, a, is obviously found by taking the cube root of a^3, the first term of the power. Subtracting a^3, we have the remainder $3a^2b + 3ab^2 + b^3$, or $(3a^2 + 3ab + b^2)b$.

The second term of the root, which is known to be b, may now be found by dividing the first term of the remainder, $3a^2b$, by $3a^2$, that is, by *three times the square of the root already found.*

Now, since $3a^2b + 3ab^2 + b^3 \equiv (3a^2 + 3ab + b^2)b$, the complete divisor is found *by adding* $3ab + b^2$ *to the trial divisor*, $3a^2$.

Multiplying the complete divisor by the second term of the root and subtracting the product from the first remainder, $3a^2b + 3ab^2 + b^3$, we have the remainder 0.

279. Since $(a + b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3 \equiv a^3 + (3a^2 + 3ab + b^2)b$ represents the perfect cube of any number, it may be used as a type, or formula, for finding the cube root of any expression, provided we always let a equal the *part of the root already found* at any stage of the process, and b *the next term to be found.*

1. Find the cube root of $8x^3 - 36x^2y + 54xy^2 - 27y^3$.

$$\begin{array}{l} 8x^3 - 36x^2y + 54xy^2 - 27y^3 \,|\, \underline{2x - 3y} \\ a^3 = (2x)^3 = \underline{8x^3} \\ 3a^2 = 3(2x)^2 = 12x^2 \quad |-36x^2y + 54xy^2 - 27y^3 \\ 3a^2 + 3ab + b^2 = 12x^2 - 18xy + 9y^2 \,|\, \underline{-36x^2y + 54xy^2 - 27y^3} = (3a^2 + 3ab + b^2)b \end{array}$$

Since $8x^3$ is the cube of the first term of the root, that term is $2x$.

The trial divisor is $3(2x)^2$, or $12x^2$. Therefore, the second term of the root is $-36x^2y \div 12x^2$, or $-3y$, and the required root is $2x - 3y$.

2. Find the cube root of $x^6 - 3x^5 + 5x^3 - 3x - 1$.

$$\begin{array}{l} x^6 - 3x^5 + 5x^3 - 3x - 1 \,|\, \underline{x^2 - x - 1} \\ \underline{x^6} \\ 3x^4 \quad |-3x^5 + 5x^3 \\ 3x^4 - 3x^3 + x^2 \,|\, \underline{-3x^5 + 3x^4 - x^3} \\ 3x^4 - 6x^3 + 3x^2 \quad |-3x^4 + 6x^3 - 3x - 1 \\ \underline{3x^4 - 6x^3 + 3x + 1} \,|\, \underline{-3x^4 + 6x^3 - 3x - 1} \end{array}$$

After the first two terms of the root have been found as in the preceding solution, the next term is obtained by treating the part of the root already found, x^2-x, as the first term of the root in finding the next term. Hence the new trial divisor is $3(x^2-x)^2$, or $3x^4-6x^3+3x^2$, and the third term of the root is -1. To form the complete divisor, we add $3(x^2-x)(-1)+1$ to the trial divisor, thus obtaining $3x^4-6x^3+3x+1$.

We then multiply the complete divisor by the last term of the root. The product subtracted from $-3x^4+6x^3-3x-1$ leaves no remainder.

$\therefore$ the required root is x^2-x-1.

Find the cube root of the following expressions:

3. $8x^3-12x^2y+6xy^2-y^3$.
4. $x^6-9x^4y^2+27x^2y^4-27y^6$.
5. $8-60a+150a^2-125a^3$.
6. $8x^6-36x^4y+54x^2y^2-27y^3$.
7. $a^3+24a^2b+192ab^2+512b^3$.
8. $x^6-6x^5+15x^4-20x^3+15x^2-6x+1$.
9. $1-6a+9a^2+4a^3-9a^4-6a^5-a^6$.
10. $64-144x+156x^2-99x^3+39x^4-9x^5+x^6$.
11. $1-6m+21m^2-44m^3+63m^4-54m^5+27m^6$.
12. $9a^3-21a^2-9a+42a^4-36a^5+8a^6-1$.
13. $8x^6+36x^5-6x^4-153x^3+15x^2+225x-125$.
14. $\frac{1}{64}x^3+\frac{3}{32}x^2y+\frac{3}{16}xy^2+\frac{1}{8}y^3$.
15. $27x^6-108x^5y+90x^4y^2+80x^3y^3-60x^2y^4-48xy^5-8y^6$.
16. $8y^6-36y^5z+114y^4z^2-207y^3z^3+285y^2z^4-225yz^5+125z^6$.
17. $\frac{x^3}{8}-\frac{3x^2}{4}+\frac{3x}{2}-1$.
18. $x^3+3x+\frac{3}{x}+\frac{1}{x^3}$.

280. Cube Roots of Arithmetical Numbers.

The method of finding the cube root of arithmetical numbers is derived from the identity

$$(t+o)^3\equiv t^3+3t^2o+3to^2+o^3\equiv t^3+(3t^2+3to+o^2)o.$$

The method is similar to that for finding the cube root of algebraic expressions, and will be understood from the following illustrative solutions:

Find the cube root of 46,656.

(a)

$$
\begin{array}{rl|l}
 & & t + o \\
 & & 46\,656 \mid 30 + 6 = 36 \\
 & & 27\,000 = t^3 \\
3\,t^2 = 3 \times 30^2 = 2700 & & 19\,656 = (3\,t^2 + 3\,to + o^2)o \\
3\,to = 3 \times 30 \times 6 = 540 & & \\
o^2 = 6^2 = 36 & & \\
\hline
3\,t^2 + 3\,to + o^2 = 3276 & & 19\,656
\end{array}
$$

(b)

$$
\begin{array}{rl}
 & 46\,656 \mid 36 \\
 & 27 \\
\hline
3 \times 30^2 = 2700 & 19\,656 \\
3 \times 30 \times 6 = 540 & \\
6^2 = 36 & \\
\hline
3276 & 19\,656
\end{array}
$$

The five figures of the number show that the root is expressed by *two* figures. Why? In 46,000 the greatest tens-cube is 27,000, and its cube root is 30, which is the *tens* of the root.

The cube of the tens, t^3, is subtracted, and the remainder, 19,656, contains $3\,t^2o + 3\,to^2 + o^3$, that is, three times the product of the square of the tens by the ones + three times the product of the tens by the square of the ones + the cube of the ones. Each of these parts contains the ones' number as a factor. Hence the 19,656 consists of two factors, one of which is the ones' number of the root; the other is three times the square of the tens + three times the product of the tens by the ones + the square of the ones. Of this last factor, three times the square of the tens, or 2700, is so much the larger part that it may be used as a *trial* divisor. Using it thus, we find 7 to be probably the ones' figure of the root. But by trial we find this value too large, and we must take one less, or 6, for the ones' figure of the root. In algebraic expressions is o definitely determined by dividing $3\,t^2o$ by $3\,t^2$? Why?

Since $3\,t^2o + 3\,to^2 + o^3$ is equal to $(3\,t^2 + 3\,to + o^2)o$, the trial divisor is completed by adding to the 2700 the $3 \times (30 \times 6) = 540$, and $6^2 = 36$; the sum, 3276, is multiplied by 6.

(1) In finding the ones' figure, we have given a product and the greater portion of one factor to find the other factor.

(2) In practice, ciphers are omitted, as in (b). The first group, 46, contains the cube of the tens' number of the root. The greatest cube in 46 is 27, and the cube root of 27 is 3. Hence 3 is the tens' figure of the root. We then divide the 196 hundreds of the remainder by the $3 \times 3^2 = 27$ hundreds to get the ones' number of the root.

281. We may apply the method given above to numbers of more than two groups of figures by always regarding *the part of the root already found as so many tens* with respect to the next figure of the root.

What is the cube root of 1,906,624?

		1 906 624 | 124
		1
$3 \times 10^2 =$	300	906
$3 \times (10 \times 2) =$	60	
$2^2 =$	4	
	364	728
$3 \times 120^2 =$	43200	178 624
$3 \times (120 \times 4) =$	1440	
$4^2 =$	16	
	44656	178 624

282. From the preceding we derive the following method of finding the cube root of arithmetical numbers:

1. *Beginning at ones, separate the number into groups of three figures each.*

2. *Find the greatest cube in the left-hand group, and write its root for the first part of the required root.*

3. *Subtract the cube of this root from the left-hand group, and to the remainder annex the next group for a dividend.*

4. *Divide this dividend by three times the square of the root already found, considered as tens. The quotient (or the quotient diminished) will be the next figure of the root.*

5. *To the last (trial) divisor add three times the product of the first part of the root, considered as tens, by the part last found, and also the square of the last part, for a complete divisor.*

6. *Multiply the complete divisor by the part of the root last found, subtract the product from the dividend, to the remainder annex the next group for a new dividend, and proceed as before, until all of the groups have been thus annexed.*

Find the cube root of the following:

1. 2744.	4. 42,875.	7. 1.191016.
2. 4096.	5. 226,981.	8. .970299.
3. 10,648.	6. 250.047.	9. $\frac{5832}{9261}$.

283. Since the fourth power may be obtained by squaring an expression and then squaring the result, and the sixth power by first cubing and then squaring the result, it follows that the fourth root is the square root of the square root, and the sixth root is the cube root of the square root.

Similarly for any other root whose index is the product of two or more numbers.

284. Broader Meaning of Root. By the ordinary definition the word "root" includes only numbers or expressions that are factors of a product of *equal* factors (Art. 265). But it is often necessary to extract the roots of numbers to a certain number of decimal places or of algebraic expressions to a certain number of terms when the numbers or expressions are not perfect powers. In such cases the meaning of the word "root" is extended to include the result obtained by extracting the root to any required number of terms by the method of extracting a root of a perfect power.

Thus, the square root of 3 to three decimal places is 1.732, and the square root of $1 + x$ to 3 terms is $\pm(1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots)$.

Find the value to three decimal places:

1. $\sqrt{2}$.	4. $\sqrt[3]{2}$.	7. $\sqrt[3]{.08}$.
2. $\sqrt{5}$.	5. $\sqrt{3.4}$.	8. $\sqrt{52.321}$.
3. $\sqrt{.4}$.	6. $\sqrt{7}$.	9. $\sqrt{\frac{5}{6}}$.

Extract the square roots of the following expressions:

10. $x^2 + 1$ to 3 terms.	13. $a^2 + 2b$ to 4 terms.
11. $a^2 + b^2$ to 4 terms.	14. $a^2 - ab + b^2$ to 4 terms.
12. $1 - 2x$ to 3 terms.	15. $1 + 2x - 3x^2$ to 4 terms.

THEORY OF EXPONENTS

285. By the ordinary definition of a power, a^m is defined as the product of m factors each equal to a. In this restricted sense there is no first power of a, because that is not the product of any number of a's, nor for a similar reason is there any zero power, negative power, or fractional power.

286. From the preceding it is evident that the original definition does not include expressions like a^1, a^0, $a^{\frac{2}{3}}$, a^{3}, $a^{-\frac{1}{2}}$, a^{-m}, etc., and hence these are as yet meaningless, except as meanings have been given to a^1 and a^0 in previous discussions.

POSITIVE INTEGRAL EXPONENTS

287. Before proceeding to discuss the meanings which should attach to *negative* and *fractional* exponents, it will be an advantage to restate the four fundamental laws which have already been proved for exponents that are *positive integers*. These are:

I.	$a^m \times a^n \equiv a^{m+n}$.	(Art. 101)
II.	$\frac{a^m}{a^n} \equiv a^{m-n}$, $m > n$.	(Art. 124)
III.	$(a^m)^n \equiv a^{mn}$.	(Art. 114)
IV.	$(ab)^m \equiv a^m b^m$.	(Art. 115)

288. Meaning of the Unit Exponent.

We have already defined a^1 to mean a (Art. 19). That this is the natural definition is seen from the fact that,

$\because$ a^2 means aa, or $a^3 \div a$,

$\therefore$ a^1 should mean a, or $a^2 \div a$.

289. Meaning of the Zero Exponent.

We have also seen that the expression a^0 may be defined to mean 1 (Art. 125). This meaning may also be derived as follows:

Assuming $a^m \times a^n \equiv a^{m+n}$ for all values of m and n, we have, when $n = 0$,

$$a^m \times a^0 \equiv a^{m+0} \equiv a^m.$$

Dividing by a^m, $$a^0 \equiv \frac{a^m}{a^m} \equiv 1.$$

That is, *any* number with zero exponent is equal to 1.

Why must $a \neq 0$ in this form? May a be either positive or negative?

NEGATIVE AND FRACTIONAL EXPONENTS

290. Since negative and fractional exponents are as yet meaningless, we assume the existence of Law I, Art. 287, for *any* rational values of the exponents, and derive consistent definitions.

291. Meaning of a Negative Integral Exponent.

I. *When the exponent is* -1.

Let $a^m \times a^n \equiv a^{m+n}$, m and n being any integers.

If $m = -1$, $n = 2$, we have

$$a^{-1} \times a^2 \equiv a^{-1+2} \equiv a.$$

Dividing by a^2, $$a^{-1} \equiv \frac{a}{a^2}$$

$$\equiv \frac{1}{a}.$$

II. *When* $m = -n$, *n being any integer.*

$$a^{-n} \times a^n \equiv a^{-n+n} \qquad \text{(Art. 290)}$$

$$\equiv a^0 \equiv 1. \qquad \text{(Art. 289)}$$

Dividing by a^n, $$a^{-n} \equiv \frac{1}{a^n}.$$

Since the assumption of a single law, viz., $a^m \times a^n \equiv a^{m+n}$, yields $a^{-n} \equiv \frac{1}{a^n}$ by a strictly logical process, we are justified in making a definition for a negative exponent as follows:

A **Negative Exponent** is a mode of expressing the reciprocal of a number with the same positive exponent.

We may therefore write

$$a^{-n} \equiv \frac{1}{a^n}, \text{ and } a^n \equiv \frac{1}{a^{-n}}.$$

One practical application of this definition is that the value of a fraction is not changed if any factor be removed from numerator to denominator, or from denominator to numerator, and the sign of its exponent be changed. [Have the student give formal proof.]

Thus, $2^{-3} = \frac{1}{2^3}$; $\frac{a^2}{a^{-3}} = a^2 \times a^3$, or a^5; $a^2x^{-3} = \frac{a^2}{x^3}$.

1. Find the value of 2^{-3}.

By definition, $$2^{-3} = \frac{1}{2^3},$$

and $$\frac{1}{2^3} = \frac{1}{8}.$$

2. Express $\frac{2^{-3}a^2b^{-3}}{3\,x^{-3}y^2}$ with positive exponents.

By definition, $$\frac{2^{-3}a^2b^{-3}}{3\,x^{-3}y^2} = \frac{a^2x^3}{3 \times 2^3b^3y^2} = \frac{a^2x^3}{24\,b^3y^2}.$$

3. Express $\frac{3\,a^3c^2}{b^{-4}d^3}$ without a denominator.

$$\frac{3\,a^3c^2}{b^{-4}d^3} = 3\,a^3b^4c^2d^{-3}.$$

Find the value of each of the following:

4. 3^{-2}.
5. $\left(\frac{1}{2}\right)^{-3}$.
6. 5^{-3}.
7. $\frac{2}{3^{-2}}$.
8. $(-2)^{-3}$.
9. $\left(\frac{1}{3}\right)^{-3}$.
10. $(.01)^{-1}$.
11. $\frac{1^{-8}}{8^{-1}}$.

Write the following expressions with positive exponents:

12. $3x^{-2}y^{-3}$.
15. $(-a)^{-3}$.
18. $ab^{-3}x^{2}y^{-2}$.
13. $\dfrac{3a^{-2}b}{cd^{-3}}$.
16. $\dfrac{a^{-2}}{4b^{-3}c}$.
19. $\dfrac{ax^{-1}z^{-2}}{5y^{-5}}$.
14. $\dfrac{a^{2}b^{-4}}{x^{-3}y^{-5}}$.
17. $\dfrac{a^{5}b^{-1}}{x^{-3}y^{2}}$.
20. $\dfrac{a^{-m}b^{n}}{x^{-m}y^{n}z^{-2}}$.

Write the following without denominators:

21. $\dfrac{a^{2}}{b^{-3}}$.
23. $\dfrac{c^{2}}{a^{3}b^{-2}}$.
25. $\dfrac{ab^{-2}}{xy^{-3}}$.
27. $\dfrac{2x^{-1}y^{5}}{3a^{-2}y^{-3}}$.
22. $\left(\dfrac{a}{b}\right)^{2}$.
24. $\dfrac{x}{a^{-3}b^{-1}}$.
26. $\dfrac{x^{-3}y^{-2}}{a^{-2}b^{-4}}$.
28. $\dfrac{4x^{-2}y^{-3}}{x^{-3}y^{-2}}$.

Write the following expressions as fractions, with positive exponents for the factors:

29. $\dfrac{y^{-2}}{x^{-2}}$.
32. $a^{m}b^{-n}c^{-p}$.
35. $3x^{-1}y^{-3}$.
30. $\dfrac{a^{2}b^{2}c^{-3}}{x^{3}y^{-2}z^{-2}}$.
33. $\dfrac{3a^{-2}c^{-3}}{5b^{-4}}$.
36. $\dfrac{a^{-2}b^{3}c^{3}}{x^{-5}y^{2}}$.
31. $\dfrac{2a^{-2}b}{xy^{-2}z^{-1}}$.
34. $\dfrac{a^{-3}b^{-2}}{c^{-5}}$.
37. $\dfrac{8a^{-3}b^{-9}}{5x^{3}y^{-2}}$.

292. Meaning of a Fractional Exponent.

I. *Consider the form* $a^{\frac{1}{2}}$.

Assuming Law I, Art. 287, as true for any rational values of m and n, and hence true for rational fractional values of the exponents, let $m = \frac{1}{2}$, $n = \frac{1}{2}$.

$$\therefore\ a^{\frac{1}{2}} \times a^{\frac{1}{2}} \equiv a^{\frac{1}{2}+\frac{1}{2}} \equiv a.$$

By original definition of exponent (Art. 19), $a^{\frac{1}{2}} \times a^{\frac{1}{2}} \equiv (a^{\frac{1}{2}})^{2}$.

Hence $(a^{\frac{1}{2}})^{2} \equiv a.$ (Axiom 1)

With the assumption of a single law for fractional exponents we find by a logical process that the form $a^{\frac{1}{2}}$ is a number whose *square* is a; hence we are justified in defining $a^{\frac{1}{2}}$ as the *square root* of a, and we have the identity

$$a^{\frac{1}{2}} \equiv \sqrt{a}.$$

II. *Consider the form* $a^{\frac{1}{n}}$, *n being a positive integer.*

Assuming the same law, $a^{\frac{1}{n}} \times a^{\frac{1}{n}} \equiv a^{\frac{2}{n}}$.

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \equiv a^{\frac{2}{n}} \times a^{\frac{1}{n}} \equiv a^{\frac{3}{n}}.$$

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \cdots \text{to } n \text{ factors} \equiv a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots \text{to } n \text{ terms}}$$

$$\equiv a^{\frac{1}{n}\times n} \equiv a.$$

By original definition of an exponent (Art. 19),

$$a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \cdots \text{to } n \text{ factors} \equiv (a^{\frac{1}{n}})^n.$$

$$\therefore (a^{\frac{1}{n}})^n \equiv a. \qquad \text{(Axiom 1)}$$

Hence we define $a^{\frac{1}{n}}$ to mean the nth root of a, and we have the identity

$$a^{\frac{1}{n}} \equiv \sqrt[n]{a}.$$

One more step is necessary for a complete knowledge of fractional exponents under the given assumption; viz.:

III. *Consider the form* $a^{\frac{m}{n}}$, *m, n being positive integers.*

By assumption of Art. 287, I, for rational fractional exponents, and its extension to any number of factors,

$$a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \cdots \text{to } n \text{ factors} \equiv a^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\cdots \text{to } n \text{ terms}}$$

$$\equiv a^{\frac{m}{n}\times n} \equiv a^m.$$

But $a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times a^{\frac{m}{n}} \times \cdots \text{to } n \text{ factors} \equiv (a^{\frac{m}{n}})^n$,

and by Axiom 1, $(a^{\frac{m}{n}})^n \equiv a^m$.

By extracting nth root of the members of the last identity,

$$a^{\frac{m}{n}} \equiv \sqrt[n]{a^m},$$

and with our assumption, we are justified in defining $a^{\frac{m}{n}}$ as the nth root of the mth power of a.

NOTE. From II, $a^{\frac{1}{n}} \equiv \sqrt[n]{a}$.

Taking mth power of the members of this identity,

$$(a^{\frac{1}{n}})^m \equiv (\sqrt[n]{a})^m.$$

By definition of exponent (Art. 19), and Art. 287, I,

$$(a^{\frac{1}{n}})^m \equiv a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times a^{\frac{1}{n}} \times \cdots \text{ to } m \text{ factors} \equiv a^{\frac{1}{n}+\frac{1}{n}+\frac{1}{n}+\cdots \text{ to } m \text{ terms}}$$

$$\equiv a^{\frac{1}{n}\times m} \equiv a^{\frac{m}{n}}.$$

By Axiom 1, $a^{\frac{m}{n}} \equiv (\sqrt[n]{a})^m$,

and from III, $(\sqrt[n]{a})^m \equiv \sqrt[n]{a^m}$.

The identity, however, is only *arithmetically* sufficient.

For example, $\sqrt{a^4} \equiv \pm a^2$; but $(\sqrt{a})^4 \equiv +a^2$. Compare Art. 296.

1. Find the value of $(-27)^{-\frac{2}{3}}$.

$$(-27)^{-\frac{2}{3}} = \frac{1}{(-27)^{\frac{2}{3}}}$$

$$= \frac{1}{(\sqrt[3]{-27})^2}$$

$$= \frac{1}{(-3)^2} = \frac{1}{9}.$$

2. Express $\dfrac{1}{\sqrt[4]{a^3}}$ in integral *form*, with fractional exponent.

$$\frac{1}{\sqrt[4]{a^3}} = \frac{1}{a^{\frac{3}{4}}} = a^{-\frac{3}{4}}.$$

Find the value of:

3. $9^{\frac{1}{2}}$.
4. $49^{-\frac{1}{2}}$.
5. $81^{\frac{3}{2}}$.
6. $1728^{\frac{2}{3}}$.
7. $16^{-\frac{3}{4}}$.
8. $(\frac{1}{25})^{-\frac{1}{2}}$.
9. $(\frac{1}{4})^{\frac{3}{2}}$.
10. $(-125)^{-\frac{4}{3}}$.

Express with positive fractional exponents:

11. $\sqrt{a^3}$.
12. $\sqrt[3]{a^3b^2}$.
13. $\sqrt[4]{a^5}\sqrt{b}$.
14. $\sqrt[3]{a^4}\sqrt[5]{x}$.
15. $3a\sqrt[m]{b}\sqrt[n]{c^m}$.
16. $\dfrac{3a\sqrt{x^3}}{4\sqrt[3]{a}\sqrt{c^3}}$.

Change from fractional exponents to radical signs:

17. $a^{\frac{2}{3}}$. 19. $x^{-\frac{1}{2}}y^{-\frac{3}{4}}$. 21. $a^{-2}b^{\frac{1}{2}}$. 23. $m^{\frac{1}{2}}n^{\frac{5}{4}}$.

18. $2a^{-\frac{1}{n}}$. 20. $3x^{\frac{1}{3}}y^{-\frac{2}{3}}$. 22. $x^{\frac{2}{3}}b^{\frac{1}{3}}$. 24. $x^{\frac{m}{n}}y^{\frac{n}{m}}$.

Change from radical signs to fractional exponents:

25. $\sqrt{a^3}$. 27. $\sqrt[3]{a^4-b^3}$. 29. $(\sqrt[n]{a-b})^m$.

26. $\sqrt[3]{a^2}$. 28. $\sqrt[3]{a^2}\sqrt[5]{b^6}$. 30. $\sqrt[n]{(a-b)x}$.

Write in integral *form*, with negative or fractional exponents:

31. $\dfrac{a^3}{c^4b^{-2}}$. 33. $\dfrac{x}{y^{-\frac{1}{4}}z^{-5}}$. 35. $\sqrt[n]{\dfrac{1}{a^m}}$.

32. $\dfrac{a^2x}{bc^{-2}}$. 34. $\dfrac{x^3y^{-\frac{1}{2}}}{m^{\frac{1}{4}}n^{-\frac{2}{3}}}$. 36. $\dfrac{1}{\sqrt[3]{(a+b)^2}}$.

LAWS FOR ANY RATIONAL EXPONENTS

293. Having found definitions for negative and fractional exponents upon the assumption of Law I, Art. 287, we shall with these definitions develop Laws II, III, and IV for any rational exponents.

294. To prove $a^m \div a^n \equiv a^{m-n}$, m and n being any rational numbers.

$$a^{m-n} \times a^n \equiv a^m. \qquad \text{(Art. 287, I)}$$

Dividing by a^n, remembering that division is the inverse of multiplication,

$$a^{m-n} \equiv a^m \div a^n.$$

1. Simplify $a^3b^{-2} \times a^{\frac{1}{3}}b^{\frac{2}{5}}$.

$$a^3b^{-2} \times a^{\frac{1}{3}}b^{\frac{2}{5}} = a^{3+\frac{1}{3}}b^{-2+\frac{2}{5}} \qquad \text{(Why?)}$$
$$= a^{\frac{10}{3}}b^{-\frac{8}{5}}.$$

2. Simplify $\dfrac{1}{\sqrt[3]{a}} \div \dfrac{1}{\sqrt[4]{a^3}}$.

$$\frac{1}{\sqrt[3]{a}} \div \frac{1}{\sqrt[4]{a^3}} = a^{-\frac{1}{3}} \div a^{-\frac{3}{4}}$$
$$= a^{-\frac{1}{3}+\frac{3}{4}}$$
$$= a^{\frac{5}{12}}, \text{ or } \sqrt[12]{a^5}.$$

Notice that this result is much more easily reached by using fractional exponents.

Simplify the following expressions:

3. $x^{\frac{1}{4}} \times x^{\frac{1}{5}}$.

4. $x^7 \times x^{-5}$.

5. $a^{-8} \div a^{-5}$.

6. $a^{\frac{1}{2}}b^{\frac{1}{3}} \times a^{\frac{1}{2}}b^{\frac{1}{3}}$.

7. $\dfrac{x^n}{x^{-n}}$.

8. $6a^{\frac{2}{3}} \div 3a^{\frac{1}{2}}$.

9. $x^{-2n} \div x^{-3n}$.

10. $a^{-\frac{1}{2}} \div a^{-\frac{1}{4}}$.

11. $a^{-\frac{2}{3}} \times a^{-\frac{1}{3}}$.

12. $(-a)^{-2} \times (-a)^{-3}$.

13. $b^{-\frac{2}{3}} \times b^{\frac{3}{4}}$.

14. $\sqrt[5]{a}\,\sqrt[3]{a^2}$.

15. $a^{\frac{1}{2}}x^{-\frac{2}{3}} \times a^{\frac{1}{2}}x^{\frac{2}{3}}$.

16. $c^{-1}d^{\frac{2}{3}} \div c^{-3}d^{\frac{1}{6}}$.

17. $2^{\frac{1}{2}}x^{\frac{1}{3}} \times 2^{\frac{3}{2}}x^{\frac{5}{3}}$.

18. $(m^{\frac{1}{3}} + n^{\frac{1}{3}})(m^{\frac{1}{3}} - n^{\frac{1}{3}})$.

19. $\dfrac{1}{\sqrt[5]{x^4}} \div \dfrac{1}{\sqrt[4]{x^3}}$.

20. $(a^{\frac{1}{4}} + 1) \times (a^{\frac{1}{4}} - 1)$.

21. $5x^{-\frac{3}{5}} \times 6\sqrt[5]{x^3}$.

22. $a\sqrt{b^{-2}} \times a^{\frac{1}{2}}\sqrt[3]{b^{-6}}$.

23. $a^{-\frac{1}{2}}b^{-\frac{3}{4}} \div a^{\frac{1}{3}}b^{-\frac{1}{2}}$.

24. $a^{\frac{1}{2}}\sqrt[3]{b^2} \times \sqrt{b^{-3}}$.

25. $(a - b) \div (a^{\frac{1}{3}} - b^{\frac{1}{3}})$.

26. $(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1) \times (a^{\frac{1}{3}} - 1)$.

27. $(a^{\frac{1}{2}} + b^{-\frac{1}{2}}) \times (a^{-\frac{1}{2}} + b^{\frac{1}{2}})$.

28. $a^{\frac{m}{n}} \div a^{-\frac{p}{q}}$.

29. $m^4n^{-\frac{5}{4}} \times m^{-4}n^{-\frac{1}{8}}$.

30. $a^{\frac{3}{4}}\sqrt{c^{-1}} \div a^{\frac{2}{3}}\sqrt[3]{c^{-4}}$.

31. $m^3\sqrt{n^3} \div m\sqrt[3]{n^2}$.

32. $a^{\frac{1}{2}}b^{\frac{1}{3}} \times a^{-1}b^{-2} \div a^2b^{-3}$.

295. To prove $(a^m)^n \equiv a^{mn}$ for all rational values of m and n.

I. Let m be any rational number and n a positive integer; then by a simple extension of Art. 287, I,

$$(a^m)^n \equiv a^m \times a^m \times a^m \cdots \text{ to } n \text{ factors} \equiv a^{m+m+m \cdots \text{ to } n \text{ terms}},$$
$$\equiv a^{mn},$$

II. Let $m = \frac{1}{q}$, $n = \frac{1}{s}$, q and s being positive integers.

Then $(a^m)^n \equiv (a^{\frac{1}{q}})^{\frac{1}{s}}$.

Now let $x = (a^{\frac{1}{q}})^{\frac{1}{s}}$.

Taking sth power, by I, $x^s = a^{\frac{1}{q}}$.

Taking qth power, similarly, $x^{sq} = a$.

Extracting sqth root of both sides, $x = \sqrt[sq]{a}$.

$$\therefore\ (a^{\frac{1}{q}})^{\frac{1}{s}} \equiv \sqrt[sq]{a} \equiv \sqrt[s]{\sqrt[q]{a}}.$$

III. Let m be a fraction $\frac{p}{q}$, and n be a fraction $\frac{r}{s}$, with p, q, r, s, positive integers.

$$(a^m)^n \equiv (a^{\frac{p}{q}})^{\frac{r}{s}} \equiv \left[\sqrt[s]{a^{\frac{p}{q}}}\right]^r \quad \text{(Def. fractional exponents)}$$

$$\equiv \left[\sqrt[s]{\sqrt[q]{a^p}}\right]^r \quad \text{(Def. fractional exponents)}$$

$$\equiv \left[\sqrt[sq]{a^p}\right]^r \quad \text{(by II)}$$

$$\equiv \left[a^{\frac{p}{sq}}\right]^r \equiv a^{\frac{pr}{sq}}. \quad \text{(Why?)}$$

The proof of these laws for negatives follows easily from the foregoing by definition of negative exponents in Art. 291.

296. Since $(a^m)^n \equiv a^{mn} \equiv (a^n)^m$ is true for rational values of m and n, it follows that $(a^m)^{\frac{1}{n}} \equiv (a^{\frac{1}{n}})^m \equiv a^{\frac{m}{n}}$.

Hence, $a^{\frac{m}{n}}$ may be considered either as the mth power of the nth root of a, or as the nth root of the mth power of a.

But this applies only to the *absolute values* of the roots, as is seen in the following:

$$(9^2)^{\frac{1}{2}} = 81^{\frac{1}{2}} = \pm 9, \text{ but } (9^{\frac{1}{2}})^2 = (\pm 3)^2 = +9.$$

Also, $(a^m)^{\frac{1}{n}} \equiv (a^{\frac{1}{n}})^{\frac{1}{m}} \equiv a^{\frac{1}{mn}}$.

That is, the mth root of the nth root of a is equal to the mnth root of a.

Have the student prove the identity $a^{\frac{m}{n}} \equiv a^{\frac{mp}{np}}$.

297. To prove $(ab)^m \equiv a^m b^m$ for any rational value of m.

It has been proved that $(ab)^m \equiv a^m b^m$ when m is a positive integer. It remains to prove the cases for a fractional or a negative exponent:

I. Let m be a fraction denoted by $\frac{p}{q}$, p and q being positive integers. We are to prove that $(ab)^{\frac{p}{q}} \equiv a^{\frac{p}{q}} b^{\frac{p}{q}}$.

Let $x = (ab)^{\frac{p}{q}}$.

Then $x^q = [(ab)^{\frac{p}{q}}]^q = (ab)^p$ (Why ?)

$= a^p b^p$.

$\therefore\ x = \sqrt[q]{a^p b^p}$ (Why ?)

$= \sqrt[q]{a^p} \times \sqrt[q]{b^p}$ (Art. 269)

$= a^{\frac{p}{q}} b^{\frac{p}{q}}$. (Definition of fractional exponents)

$\therefore\ (ab)^{\frac{p}{q}} \equiv a^{\frac{p}{q}} b^{\frac{p}{q}}$. (Ax. 1)

II. Let $m = -r$, r being a positive integer or fraction.

Then $(ab)^{-r} \equiv \frac{1}{(ab)^r}$ (Art. 291)

$\equiv \frac{1}{a^r b^r}$, (by I)

$\equiv a^{-r} b^{-r}$. (Why ?)

As an exercise the student may prove that $\left(\frac{a}{b}\right)^n \equiv \frac{a^n}{b^n}$ for all values of n.

NOTE. It is now obvious that roots of numbers may be indicated either by exponents or by the radical sign. It will also be seen that for some purposes the notation by exponents is the more convenient. It is especially useful in facilitating many operations.

1. Simplify $(x^{-2})^{-3}$.

$$(x^{-2})^{-3} = \left(\frac{1}{x}\right)^{-3} = \frac{1}{\left(\frac{1}{x^2}\right)^3} = x^6.$$

2. Simplify $\sqrt{(9\,a^{-\frac{2}{3}})^3}$, or $(9\,a^{-\frac{2}{3}})^{\frac{3}{2}}$.

$$(9\,a^{-\frac{2}{3}})^{\frac{3}{2}} = (9^3\,a^{-2})^{\frac{1}{2}} = 9^{\frac{3}{2}}\,a^{-1} = \frac{27}{a}.$$

3. Simplify $\left(\frac{81\,m^{-8}}{16\,n^3}\right)^{-\frac{3}{4}}$.

$$\left(\frac{81\,m^{-8}}{16\,n^3}\right)^{-\frac{3}{4}} = \left(\frac{16\,n^3}{81\,m^{-8}}\right)^{\frac{3}{4}} = \left(\frac{2\,m^2n^{\frac{3}{4}}}{3}\right)^3 = \frac{8\,m^6n^{\frac{9}{4}}}{27}.$$

Simplify the following expressions:

4. $(a^3)^{\frac{1}{2}}$.
5. $(a^2)^{-3}$.
6. $(a^{\frac{1}{3}})^{-3}$.
7. $(a^5)^{-2}$.
8. $(ab)^{-4}$.
9. $(2\,x^{-2}y)^{-3}$.
10. $(-\frac{1}{3^2}\,xy)^{-\frac{2}{3}}$.
11. $(x^{-2})^{-2}$.
12. $(3\,x)^{-3}$.
13. $(a^5b^{\frac{1}{2}})^{\frac{2}{5}}$.
14. $\left(\frac{a^3}{b^9}\right)^{-\frac{1}{2}}$.
15. $(x^{-\frac{1}{2}}y^2)^{-4}$.
16. $(\sqrt{a^3})^{-3}$.
17. $(5\,x^2y^{-3})^{\frac{m}{n}}$.
18. $(a^2b^{-\frac{1}{2}})^{-\frac{1}{4}}$.
19. $\sqrt[6]{(x^{-1})^3}$.
20. $\sqrt[3]{(8\,a^{-\frac{5}{6}})^2}$.
21. $(x^3y^{-\frac{2}{3}})^3$.
22. $(-5\,x^{\frac{1}{3}}y^{\frac{1}{2}})^{-2}$.
23. $(-3\,a^{-\frac{1}{3}}b^{\frac{1}{2}})^{-\frac{3}{4}}$.
24. $(a^3)^{-\frac{1}{6}} \times (a^{-1})^{-2}$.

MISCELLANEOUS EXERCISES

298. 1. Simplify $(\sqrt[3]{-8\,a^{-2}})^{-2}$.

$$(\sqrt[3]{-8\,a^{-2}})^{-2} = \frac{1}{(\sqrt[3]{-8\,a^{-2}})^2} = \frac{1}{\left(\sqrt[3]{\frac{a^2}{-8}}\right)^2} = \frac{1}{\left(\frac{a^2}{-8}\right)^{\frac{2}{3}}}$$

$$= \frac{1}{\frac{a^{\frac{4}{3}}}{-8^{\frac{2}{3}}}} = \frac{1}{\frac{a^{\frac{4}{3}}}{4}} = \frac{4}{a^{\frac{4}{3}}}.$$

2. Multiply $x^{\frac{1}{3}} - 2x^{\frac{1}{3}}y^{-\frac{1}{6}} + y^{-\frac{1}{3}}$ by $x^{\frac{1}{6}} - y^{-\frac{1}{6}}$.

$$
\begin{array}{l}
x^{\frac{1}{3}} - 2x^{\frac{1}{6}}y^{-\frac{1}{6}} + y^{-\frac{1}{3}} \\
x^{\frac{1}{6}} - y^{-\frac{1}{6}} \\
\hline
x^{\frac{1}{2}} - 2x^{\frac{1}{3}}y^{-\frac{1}{6}} + \quad x^{\frac{1}{6}}y^{-\frac{1}{3}} \\
\quad - \quad x^{\frac{1}{3}}y^{-\frac{1}{6}} + 2x^{\frac{1}{6}}y^{-\frac{1}{3}} - y^{-\frac{1}{2}} \\
\hline
x^{\frac{1}{2}} - 3x^{\frac{1}{3}}y^{-\frac{1}{6}} + 3x^{\frac{1}{6}}y^{-\frac{1}{3}} - y^{-\frac{1}{2}}
\end{array}
$$

3. Divide $x^{-3} + 3x^{-2} + 3x^{-1} + 1$ by $x^{-2} + 2x^{-1} + 1$.

$$
\begin{array}{l|l}
x^{-3} + 3x^{-2} + 3x^{-1} + 1 & x^{-2} + 2x^{-1} + 1 \\
\underline{x^{-3} + 2x^{-2} + \quad x^{-1}} & x^{-1} + 1 \\
\quad x^{-2} + 2x^{-1} + 1 & \\
\quad \underline{x^{-2} + 2x^{-1} + 1} &
\end{array}
$$

Find the numerical value of:

4. $100^{-\frac{3}{2}}$.

5. $1000^{\frac{2}{3}}$.

6. $16^{-\frac{3}{2}}$.

7. $4^{-\frac{5}{2}}$.

8. $(\frac{4}{5})^{\frac{1}{2}} \times (3\frac{1}{3})^{\frac{1}{2}} \times (\frac{3}{2})^{\frac{1}{2}}$.

9. $(7\frac{1}{2})^{\frac{1}{3}} \times (2\frac{2}{3})^{\frac{1}{3}} \times (3\frac{1}{5})^{\frac{1}{3}}$.

Express with positive exponents:

10. $\dfrac{2}{x^{-2}}$.

11. $\sqrt[4]{a^{-3}}$.

12. $\dfrac{x^{-m}z^{3}}{y^{-n}z^{-1}}$.

13. $5x^{-3}y^{-2}$.

14. $\dfrac{2^{-3}x^{-2}y^{2}}{16a^{-3}b^{2}}$.

15. $\dfrac{\sqrt[4]{a^{7}}}{\sqrt{a^{-3}}}$.

16. $\dfrac{a^{\frac{1}{2}}b^{2}x^{-\frac{1}{3}}}{a^{\frac{1}{2}}b^{-2}x^{-\frac{2}{3}}}$.

17. $\dfrac{5^{-2}x^{\frac{3}{5}}y^{-\frac{2}{3}}}{x^{-\frac{3}{5}}y^{\frac{2}{3}}}$.

18. $\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$.

Express by radical signs and positive exponents:

19. $x^{-\frac{2}{3}}$.

20. $\dfrac{a^{-\frac{1}{3}}}{a^{-2}}$.

21. $\dfrac{1}{a^{\frac{-1}{n}}}$.

22. $a^{\frac{1}{2}} - x^{\frac{1}{3}}c^{\frac{1}{3}}$.

23. $a^{\frac{3}{4}}b^{-2} - a^{\frac{3}{4}}b^{2}$.

24. $x^{-\frac{3}{2}}y(xy^{-2})^{-\frac{1}{2}}$.

Express with fractional exponents:

25. $\sqrt[3]{b^4}$.
26. $(\sqrt{a})^6$.
27. $\sqrt[3]{a^5}$.
28. $\sqrt[3]{-8}$.
29. $\sqrt[n]{x^m}$.
30. $\sqrt[3]{8\,a^{-2}b^{\frac{2}{3}}c^{-1}}$.
31. $\sqrt[4]{(x-y)^3}$.
32. $\sqrt[n]{ax}$.
33. $\sqrt[3]{(x^{-1}y^{-2})^{-\frac{3}{4}}}$.

Express without denominators:

34. $\dfrac{3\,x^2}{2\,y^{-2}}$.
35. $\dfrac{x^{-\frac{1}{2}}y^{\frac{2}{3}}}{a^{-2}y^{-\frac{4}{3}}}$.
36. $\dfrac{a^{-1}b^{-2}c^{-3}}{a^{-2}b^{-2}c^{-4}}$.
37. $\dfrac{ab^{\frac{1}{2}}}{2^{-3}a^{-3}b^{-\frac{1}{2}}}$.
38. $\dfrac{3\,a^{\frac{1}{3}}b^{-3}}{3^{-1}a^{-\frac{2}{3}}b^3}$.
39. $\dfrac{1}{x}+\dfrac{2}{x^2}+\dfrac{3}{x^3}$.

Multiply:

40. $a^{\frac{1}{2}}+a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$ by $a^{\frac{1}{2}}-a^{\frac{1}{4}}b^{\frac{1}{4}}+b^{\frac{1}{2}}$.
41. $x^{\frac{4}{3}}+x^{\frac{2}{3}}+1$ by $x^{\frac{2}{3}}+x-x^{\frac{1}{3}}$.
42. $2\,x^{\frac{2}{3}}-3\,x^{\frac{1}{3}}-4+x^{-\frac{1}{3}}$ by $3\,x^{\frac{4}{3}}+x-2\,x^{\frac{2}{3}}$.
43. $a+3\,a^{\frac{2}{3}}-2\,a^{\frac{1}{3}}$ by $3-2\,a^{-\frac{1}{3}}+4\,a^{-\frac{2}{3}}$.
44. $(a^{-1}+b^{-1}+c^{-1})(a^{-1}+b^{-1}-c^{-1})$.
45. $4\,a^{-1}-5\,x^{-1}+6\,ax^{-2}$ by $a^2x^{-1}+3\,a^3x^{-2}$.
46. $m+3\,m^{\frac{1}{3}}-4\,m^{\frac{2}{3}}$ by $3-5\,m^{-\frac{2}{3}}-7\,m^{-\frac{1}{3}}$.
47. $x^n+x^{\frac{1}{2}n}+1$ by $x^{-n}-x^{-\frac{1}{2}n}+1$.
48. $x^2+2\,xy+y^2$ by $x^{-2}-2\,x^{-1}y^{-1}+y^{-2}$.
49. $x^3-x^{-3}-2\,x^{-6}+5$ by $10\,x^{-7}+x^{-1}-5\,x^{-4}$.
50. $a^{-\frac{1}{2}}\sqrt{b}+2-\sqrt{ab^{-1}}$ by $\sqrt{a^{-1}}-\sqrt{b^{-1}}$.
51. $x^{\frac{1}{3}}y^{-\frac{1}{2}}+1+x^{-\frac{1}{3}}y^{\frac{1}{2}}$ by $x^{\frac{1}{3}}y^{-\frac{1}{2}}-1+x^{-\frac{1}{3}}y^{\frac{1}{2}}$.

Divide:

52. $x^{\frac{2}{3}}+x^{\frac{1}{3}}-12$ by $x^{\frac{1}{3}}-3$.
53. $a^{-3}+2\,a^{-2}b^{-1}-3\,b^{-3}$ by $a^{-1}-b^{-1}$.
54. $a^{\frac{3}{2}}-ab^{\frac{1}{2}}+a^{\frac{1}{2}}b-b^{\frac{3}{2}}$ by $a^{\frac{1}{2}}-b^{\frac{1}{2}}$.
55. $x^{-\frac{3}{2}}-x^{-1}y^{\frac{1}{2}}+x^{-\frac{1}{2}}y-y^{\frac{3}{2}}$ by $x^{-1}+y$.

56. $a^{\frac{2}{3}} + a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{2}{3}}$ by $a^{\frac{1}{3}} - a^{\frac{1}{3}}b^{\frac{1}{3}} + b^{\frac{1}{3}}$.

57. $x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y$ by $x^{\frac{1}{2}} + x^{\frac{1}{4}}y^{\frac{1}{4}} + y^{\frac{1}{4}}$.

58. $8\,a^{-n} - 8\,a^{n} + 5\,a^{3n} - 3\,a^{-3n}$ by $5\,a^{n} - 3\,a^{-n}$.

59. $16\,a^{-3} - 6\,a^{-2} + 5\,a^{-1} + 6$ by $2\,a^{-1} + 1$.

60. $\sqrt{a} + \sqrt[4]{ab} + \sqrt{b}$ by $\sqrt[4]{a} + \sqrt[8]{ab} + \sqrt[4]{b}$.

61. $x + y - 3\,x^{\frac{1}{3}}y^{\frac{1}{3}}z^{\frac{1}{3}} + z$ by $x^{\frac{1}{3}} + y^{\frac{1}{3}} + z^{\frac{1}{3}}$.

Simplify each of the following expressions:

62. $(x^{\frac{1}{2}}y^{\frac{3}{2}})^{\frac{1}{3}} \div (x^{-\frac{1}{3}}y)^{\frac{1}{2}}$.

63. $(a^{-\frac{1}{3}}b^{-3})^{-4}$.

64. $(a^{-1})^{-2} \times (a^{-\frac{1}{2}})^{2}$.

65. $(\sqrt{a})^{3}\sqrt[3]{a^{-4}}$.

66. $\dfrac{\sqrt{x^{3}} \times \sqrt[3]{y^{2}}}{\sqrt[6]{y^{-2}} \times \sqrt[4]{x^{6}}}$.

67. $\dfrac{n(rx)^{\frac{1}{2}}(nbx)^{2}}{(ax)^{\frac{1}{2}}\sqrt{nra}}$.

68. $x^{\frac{1}{3}}y^{\frac{2}{3}} \div x^{-\frac{2}{3}}y^{\frac{4}{3}}$.

69. $\left(\sqrt{(a+x)}\sqrt{x}\right)^{4}$.

70. $\left(3^{n} - 2 + \dfrac{1}{3^{n}}\right)^{\frac{1}{2}}$.

71. $[-(a^{3})^{\frac{1}{2}}]^{-\frac{1}{2}}$.

72. $\dfrac{\sqrt[3]{x^{-1}\sqrt{y^{3}}}}{\sqrt{y^{-1}\sqrt[3]{x^{2}}}}$.

73. $\left(\dfrac{a^{-\frac{2}{3}}x^{\frac{1}{2}}}{x^{-1}a}\right)^{2} \div \sqrt[3]{\dfrac{a^{-1}}{x^{-3}}}$.

Find the square root of each of the following expressions:

74. $x^{\frac{2}{3}} + 2\,x^{\frac{1}{3}} + 1$.

75. $9\,xy^{-2} + 12\,y^{-1} + 4\,x^{-1}$.

76. $9\,a^{2} - 12 + 10\,a^{-2} - 4\,a^{-4} + a^{-6}$.

77. $4\,a^{-2}x^{2} - 12\,a^{-1}x + 25 - 24\,ax^{-1} + 16\,a^{2}x^{-2}$.

78. $1 + 4\,a^{-\frac{1}{3}} - 2\,a^{-\frac{2}{3}} - 4\,a^{-1} + 25\,a^{-\frac{4}{3}} - 24\,a^{-\frac{5}{3}} + 16\,a^{-2}$.

Find the cube root of each of the following expressions:

79. $x^{\frac{3}{5}} - 3\,x^{\frac{2}{5}} + 3\,x^{\frac{1}{5}} - 1$.

80. $\frac{1}{27}\,x^{-6} + \frac{2}{3}\,x^{-4} + 4\,x^{-2} + 8$.

81. $a^{-\frac{3}{4}} - 3\,a^{-1}b^{\frac{2}{3}} + 3\,a^{-\frac{1}{2}}b^{\frac{4}{3}} - b^{2}$.

RADICALS AND SURDS

299. A **Rational Number** is a number expressible either as an integer or as a fraction whose numerator and denominator are integers. The term is seen to include positive and negative integers and fractions.

All other numbers are said to be **Irrational.**

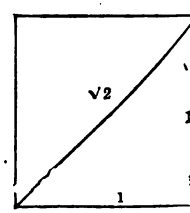
Thus, 2, $\frac{2}{3}$, $.333\cdots$ are rational numbers, while $\sqrt{2}$, $(2+\sqrt{3})^{\frac{1}{2}}$, $\sqrt{-5}$ are irrational numbers.

300. Irrational roots of rational numbers (excepting even roots of negative numbers) are called **Surds.**

Thus, $\sqrt{5}$ is a surd, since the square root of 5 is irrational; but $\sqrt{2+\sqrt{2}}$ is not a surd, since $2+\sqrt{2}$ is not a rational number.

Observe that while every surd is an irrational number there are irrational numbers which are not surds, as $3.1415\cdots$, $\sqrt{-1}$, etc. It is also to be observed that an algebraic expression that is irrational may or may not be irrational when numerical values are assigned. For instance, $\sqrt{x}$ becomes rational when $x = 4$.

301. By the introduction of surds we make possible a formal solution of the equation $x^2 - 2 = 0$, which is satisfied by $\sqrt{2}$. The surd $\sqrt{2}$ finds a concrete representation in the diagonal of a square whose side is one unit in length, as shown in the annexed figure.



302. Any number or expression which involves a radical symbol (radical sign or fractional exponent) is called a **Radical.**

Thus, $\sqrt{3}$, $\sqrt{4}$, $\sqrt[3]{7}$, $\sqrt[3]{a^6}$, $(a+b)^{\frac{1}{2}}$ are radicals. It is seen that radicals may be rational or irrational, but surds are always irrational.

303. Order of Surds. A surd is said to be of the second, third, or nth orders according as the index of the root is 2, 3, or n. Surds of the second order are also called *quadratic* surds.

Thus, $\sqrt{3}$ is a quadratic surd, or surd of the second order, $\sqrt[3]{5}$ is a surd of the third order, and so on.

304. Surds composed of a rational and a surd factor are called **Mixed Surds**; those which have no rational factor are called **Pure Surds.**

Thus, $\sqrt{3}$ is a pure surd; $2\sqrt{3}$ is a mixed surd.

305. A surd is said to be in its *simplest form* when the root index is as small as possible, and the number included under the radical symbol is integral and as small as possible.

Thus, $\sqrt{2}$ is in the simplest form, but $\sqrt{\frac{1}{2}}$ is not, since $\sqrt{\frac{1}{2}} = \sqrt{\frac{2}{4}} = \sqrt{\frac{1}{4} \times 2} = \frac{1}{2}\sqrt{2}$; $\sqrt{8}$ is not in its simplest form, since $\sqrt{8} = \sqrt{4 \times 2} = 2\sqrt{2}$.

306. Two or more surds are said to be **similar** when, in their simplest form, they have the same surd factor.

Thus, $2\sqrt{2}$ and $3\sqrt{2}$ are similar; $\sqrt{8}$ and $\sqrt{18}$ are considered similar, since in simplest form they are equal to $2\sqrt{2}$ and $3\sqrt{2}$ respectively.

307. It is evident that the radical expression $\sqrt{9} + \sqrt{16}$ reduces to $(\pm 3) + (\pm 4)$; that is, to $3 + 4 = 7$, $3 - 4 = -1$, $-3 + 4 = 1$, or $-3 - 4 = -7$. However, for simplicity it is customary to consider only the *positive values* of expressions of this kind, unless the $\pm$ is expressed. In this restricted sense $\sqrt{9} + \sqrt{16}$ means $3 + 4$, or 7.

REDUCTION OF SURDS

308. Since $a \equiv (a^n)^{\frac{1}{n}}$, it follows that a rational number can be reduced to the *form* of a surd of any given order.

1. Reduce 3 to the form of a surd of the second order.

$$3 = (3^2)^{\frac{1}{2}} = \sqrt{9}.$$

2. Reduce 2 to the form of a surd of the 4th order.

$$2 = (2^4)^{\frac{1}{4}} = \sqrt[4]{16}.$$

309. Since $a\sqrt[n]{b} \equiv \sqrt[n]{a^n b}$, it follows that mixed surds can always be reduced to the form of pure surds.

1. Reduce $2\sqrt{3}$ to the form of a pure surd.

$$2\sqrt{3} = \sqrt{2^2 \times 3} = \sqrt{12}.$$

Reduce to the form of surds of the orders indicated:

2. 2, 3d order.
3. -3, 3d order.
4. 3, 5th order.
5. m, 2d order.
6. a, 3d order.
7. x, 6th order.
8. $3ab$, 2d order.
9. $a+x$, nth order.
10. $(a+b)^{\frac{1}{2}}$, 4th order.

Reduce to the form of pure surds:

11. $3\sqrt{2}$.
12. $-2\sqrt{3}$.
13. $-3\sqrt[3]{2}$.
14. $-3\sqrt[3]{5}$.
15. $\frac{1}{2}\sqrt{8}$.
16. $\frac{m}{n}\sqrt{\frac{n}{m}}$.
17. $ab\sqrt{c}$.
18. $3x\sqrt{xy}$.
19. $2a\sqrt[4]{5}$.
22. $(x-y)\sqrt{\frac{1}{x-y}}$.
23. $ax^{\frac{1}{2}}$.
24. $2 \times 9^{\frac{1}{3}}$.
25. $3ab^2\sqrt[5]{c^3}$.
26. $(m-n)\sqrt[n]{x}$.
27. $(a+b)\sqrt{c}$.
28. $\frac{a+b}{a-b}\sqrt{\frac{a-b}{a+b}}$.

310. Reduction to Simplest Forms.

In the reduction of surds to their simplest forms two important cases arise. The first involves the removal of a rational factor from under the radical sign, being the reverse of the process of reducing a mixed surd to the form of a pure surd; and the second involves the removal of the denominator when the number under the radical sign is a fraction. In the latter case, *we multiply the numerator and denominator of the fraction by the smallest number which will make the denominator a power with an exponent equal to the index of the root, then extract the indicated root of the denominator, and simplify the remaining surd as much as possible.*

1. Reduce $\sqrt[3]{54}$ to its simplest form.

Evidently $\sqrt[3]{54} = \sqrt[3]{3^3 \times 2} = 3\sqrt[3]{2}$.

2. Reduce $\sqrt{\frac{3}{50}}$ to its simplest form.

Here $\sqrt{\frac{3}{50}} = \sqrt{\frac{3}{5^2 \times 2}} = \sqrt{\frac{3 \times 2}{5^2 \times 2^2}} = \frac{1}{10}\sqrt{6}$.

3. Reduce $\sqrt[9]{27}$ to its simplest form.

$$\sqrt[9]{27} = (3^3)^{\frac{1}{9}} = 3^{3\times\frac{1}{9}} \qquad \text{(Art. 295)}$$
$$= 3^{\frac{1}{3}} = \sqrt[3]{3}.$$

Reduce to their simplest forms:

4. $\sqrt{20}$.

5. $5\sqrt{\frac{1}{3}}$.

6. $\sqrt[4]{9}$.

7. $\sqrt{192}$.

8. $\sqrt[10]{32}$.

9. $\sqrt[5]{160}$.

10. $\sqrt[4]{162}$.

11. $\sqrt{432}$.

12. $\sqrt[9]{64}$.

13. $\sqrt[3]{108}$.

14. $\sqrt[4]{144}$.

15. $\sqrt[3]{135}$.

16. $\sqrt[6]{27}$.

17. $\sqrt[3]{-32}$.

18. $(1\frac{1}{8})^{-\frac{1}{2}}$.

19. $\sqrt[6]{1728}$.

20. $\sqrt{3\sqrt{3}}$.

21. $2\sqrt[4]{80}$.

22. $\sqrt[3]{27\,a^3x}$.

23. $\sqrt{18\,x^3y^5}$.

24. $\sqrt[8]{81}$.

25. $\sqrt[12]{a^9b^6}$.

26. $\sqrt[3]{54\,a^2b^3}$.

27. $\frac{1}{8}\sqrt{176}$.

28. $\sqrt{48\,a^2b}$.

29. $\sqrt{8\,a^2bc}$.

30. $\sqrt{a^4b^5}$.

31. $\sqrt[3]{64\,a^6b^2}$.

32. $\sqrt{20\,a^2 - 8}$.

33. $\sqrt[4]{(a^2 - b^2)^2}$.

34. $7\sqrt[3]{-108\,a^6b^7c^9}$.

35. $\sqrt{16a^2 - 16a^2x^2}$.

36. $\sqrt{(x^2-1)(x+1)}$.

37. $\sqrt{\frac{2}{3}}$.

38. $\sqrt[3]{\frac{1}{2}}$.

39. $\sqrt[4]{\frac{7}{8}}$.

40. $\frac{1}{2}\sqrt{\frac{1}{3}}$.

41. $\sqrt{\frac{5}{12}}$.

42. $\sqrt[3]{\frac{1}{4}}$.

43. $\sqrt{\frac{25}{24}}$.

44. $\sqrt[3]{\frac{7}{12}}$.

45. $\sqrt[5]{\frac{2}{3}}$.

46. $\sqrt[3]{\frac{5}{9}}$.

47. $\sqrt[3]{\frac{3}{8}}$.

48. $\sqrt{\frac{a^2}{2\,b^3}}$.

49. $\sqrt[4]{\frac{a}{b}}$.

50. $\sqrt{\frac{a}{7}}$.

51. $\sqrt[3]{\frac{3}{25}}$.

52. $\sqrt[3]{\frac{4}{9}\frac{ab}{c^2}}$.

311. Reduction to Identical Surds of the Same Order.

Since $a^{\frac{m}{n}} \equiv a^{\frac{mp}{np}}$, it follows that surds of different orders can be reduced to identical surds of the same order *by expressing them with fractional exponents, and multiplying both numerator and denominator of each fractional exponent by the quotient obtained from dividing the least common multiple of the denominators of all the exponents by the denominator of that exponent.*

1. Reduce $\sqrt{5}$ and $\sqrt[3]{3}$ to identical surds of the same order.

The L. C. M. of 2 and 3 is 6.

$$\therefore 5^{\frac{1}{2}} = 5^{\frac{3}{6}} = \sqrt[6]{5^3}, \text{ or } \sqrt[6]{125};$$
$$3^{\frac{1}{3}} = 3^{\frac{2}{6}} = \sqrt[6]{3^2}, \text{ or } \sqrt[6]{9}.$$

2. Reduce $\sqrt[3]{3}$ and $\sqrt[4]{5}$ to identical surds of the same order.

Here
$$3^{\frac{1}{3}} = 3^{\frac{4}{12}} = \sqrt[12]{3^4}, \text{ or } \sqrt[12]{81};$$
$$5^{\frac{1}{4}} = 5^{\frac{3}{12}} = \sqrt[12]{5^3}, \text{ or } \sqrt[12]{125}.$$

Reduce to identical surds of same order:

3. $\sqrt{2}$, $\sqrt[3]{3}$.
4. $\sqrt[3]{4}$, $\sqrt[4]{3}$.
5. $\sqrt{a}$, $\sqrt[3]{b}$.
6. $\sqrt[4]{24}$, $\sqrt[3]{11}$.
7. $\sqrt[4]{3}$, $\sqrt[6]{5}$.
8. $\sqrt{a}$, $\sqrt[m]{b}$.
9. $\sqrt{\frac{1}{2}}$, $\sqrt[3]{\frac{1}{4}}$.
10. $\sqrt{a+b}$, $\sqrt[3]{m+n}$.
11. $\sqrt[4]{1-a}$, $\sqrt[6]{1+a}$.
12. $3^{\frac{1}{2}}$, $3^{\frac{3}{4}}$, $3^{\frac{2}{3}}$.
13. $\sqrt{2}$, $\sqrt[3]{3}$, $\sqrt[4]{5}$.
14. $\sqrt{ab}$, $\sqrt[4]{bc}$, $\sqrt[3]{ac}$.
15. $\sqrt{2}$, $\sqrt[4]{2x}$, $\sqrt[6]{5y}$.
16. $\sqrt[4]{2}$, $\sqrt{3}$, $\sqrt[8]{5}$.
17. $\sqrt[3]{2}$, $\sqrt[9]{9}$, $\sqrt[6]{5}$.

ADDITION AND SUBTRACTION OF SURDS

312. It is necessary that surds be of the same order and have a common surd factor before they can be added or subtracted.

Hence, to add surds *reduce them to their simplest forms, and add the rational factors of the resulting surds which are similar, writing those which are not similar after one another with their proper signs.*

A similar method applies in subtracting surds.

1. Find the sum of $3\sqrt{2}$, $\sqrt{18}$, $2\sqrt{32}$.

$$3\sqrt{2} = 3\sqrt{2}.$$
$$\sqrt{18} = \sqrt{9 \times 2} = 3\sqrt{2}.$$
$$2\sqrt{32} = 2\sqrt{16 \times 2} = 8\sqrt{2}.$$
$$\therefore \text{ the sum} = 14\sqrt{2}.$$

2. Simplify $\sqrt{27} + \sqrt{48} - \sqrt{75}$.

$$\sqrt{27} = \sqrt{9 \times 3} = 3\sqrt{3}.$$
$$\sqrt{48} = \sqrt{16 \times 3} = 4\sqrt{3}.$$
$$\sqrt{75} = \sqrt{25 \times 3} = 5\sqrt{3}.$$
$$\therefore \sqrt{27} + \sqrt{48} - \sqrt{75} = 3\sqrt{3} + 4\sqrt{3} - 5\sqrt{3} = 2\sqrt{3}.$$

Find the sum of:

3. $3\sqrt{45}$, $\sqrt{20}$.

4. $2\sqrt{28}$, $\sqrt{63}$.

5. $5\sqrt{2}$, $\sqrt{128}$.

6. $\sqrt{27}$, $\sqrt{48}$, $\sqrt{147}$.

7. $\sqrt{8}$, $\sqrt{18}$, $\sqrt{32}$.

8. $2\sqrt{3}$, $\frac{1}{2}\sqrt{12}$, $4\sqrt{27}$.

Simplify the following expressions:

9. $\sqrt{24} + \sqrt{54} - \sqrt{96}$.

10. $\sqrt{12} + 3\sqrt{75} - 2\sqrt{27}$.

11. $3\sqrt{3} + \sqrt{27} - 11\sqrt{48}$.

12. $5\sqrt{24} - 2\sqrt{54} + \sqrt{6}$.

13. $\sqrt{27} + 3\sqrt{108} + 2\sqrt{75}$.

14. $\sqrt{\frac{8}{5}} + \sqrt{\frac{9}{10}} - \sqrt{\frac{5}{8}}$.

15. $\sqrt{\frac{3}{4}} + \sqrt{\frac{1}{3}} - \frac{1}{4}\sqrt{3}$.

16. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{3}} - \sqrt{1\frac{1}{8}}$.

17. $\sqrt[4]{243} + \sqrt[4]{48} - \sqrt[4]{768}$.

18. $\sqrt{50} + 2\sqrt[6]{8} + 6\sqrt{\frac{1}{2}}$.

19. $5\sqrt[3]{4} + 6\sqrt[3]{32} - 5\sqrt[3]{108}$.

20. $\sqrt{x} + \frac{1}{2}\sqrt{x} + \frac{3}{2}\sqrt{x}$.

21. $\sqrt{2} + \sqrt{18} - \sqrt{50} + \sqrt{162}$.

22. $\sqrt{243} + \sqrt{75} - \sqrt{12} - \sqrt{\frac{1}{3}}$.

23. $6a\sqrt{63ab^3} - 3\sqrt{112a^3b^3}$.

24. $\sqrt{a^5b} + 2\sqrt{a^3b^3} + \sqrt{ab^5}$.

25. $\sqrt{50} + \sqrt{54} - \sqrt{24} + 2\sqrt{\frac{3}{2}}$.

26. $a\sqrt[3]{b} + a^2\sqrt[3]{b^4} - a^3\sqrt[3]{b^7}$.

27. $\sqrt{8(x-y)} + 4\sqrt{18(x-y)} - 12\sqrt{\frac{1}{8}(x-y)}$.

MULTIPLICATION OF SURDS

313. Since $\sqrt[n]{a} \times \sqrt[n]{b} \equiv \sqrt[n]{ab}$ (Art. 269), it follows that the product of two or more surd factors may be found *by first reducing the surds of different orders to the same order, and then taking the product of the rational factors for one factor of the product and the product of the surd factors for its other factor, reducing the result to its simplest form.*

The arrangement of the work in the multiplication of polynomial surds is illustrated in the solutions of 3 and 4 below.

1. Find the product of $2\sqrt{3}$ and $3\sqrt{5}$.

$$2\sqrt{3} \times 3\sqrt{5} = (2 \times 3)(\sqrt{3} \times \sqrt{5}) = 6\sqrt{3 \times 5} = 6\sqrt{15}.$$

2. Find the product of $\sqrt[3]{2}$ and $\sqrt{3}$.

$$\begin{aligned} 2^{\frac{1}{3}} \times 3^{\frac{1}{2}} &= 2^{\frac{2}{6}} \times 3^{\frac{3}{6}} \\ &= (2^2 \times 3^3)^{\frac{1}{6}} \\ &= \sqrt[6]{4 \times 27}, \text{ or } \sqrt[6]{108}. \end{aligned}$$

3. Find the product of $5 + 2\sqrt{5}$ by $5 - 2\sqrt{5}$.

$$\begin{array}{r} 5 + 2\sqrt{5} \\ 5 - 2\sqrt{5} \\ \hline 25 + 10\sqrt{5} \\ \quad - 10\sqrt{5} - 4\sqrt{5^2} \\ \hline 25 \qquad\quad - 20 = 5 \end{array}$$

4. Square $\sqrt{3} - 2\sqrt{5}$.

Since $(a - b)^2 \equiv a^2 - 2ab + b^2$,

$$\begin{aligned} (\sqrt{3} - 2\sqrt{5})^2 &= (\sqrt{3})^2 - 2(\sqrt{3})(2\sqrt{5}) + (2\sqrt{5})^2 \\ &= 3 - 4\sqrt{15} + 20 \\ &= 23 - 4\sqrt{15}. \end{aligned}$$

Many products are more readily obtained by using the type forms for rational expressions. In Ex. 3 the product may be found by the use of the type form for the *product of the sum and difference of two numbers.*

Find the product of:

5. $\sqrt{3}$ by $\sqrt{2}$.
6. $\sqrt{63}$ by $\sqrt{7}$.
7. $2\sqrt{3}$ by $3\sqrt[3]{2}$.
8. $\sqrt[3]{6}$ by $3\sqrt[3]{36}$.
9. $\sqrt[4]{5}$ by $\sqrt[6]{10}$.
10. $\frac{1}{2}\sqrt[3]{4}$ by $2\sqrt[3]{2}$.
11. $\sqrt[3]{\frac{2}{3}}$ by $\sqrt[5]{\frac{1}{2}}$.
12. $\sqrt{10}$ by $\sqrt[3]{200}$.
13. $\sqrt{2xy}$ by $\sqrt[3]{xyz}$.
14. $\sqrt{m}$ by $\sqrt[3]{m^2}$.
15. $a\sqrt{b^3}$ by $b^2\sqrt{a}$.
16. $\sqrt[m]{a}$ by $\sqrt[n]{b}$.
17. $3a\sqrt{b^3}$ by $2b\sqrt[3]{a^2}$.
18. $\sqrt{3}$ by $\sqrt[3]{9}$.
19. $8\sqrt{3}$ by $2\sqrt[3]{5}$.
20. $\sqrt[3]{2}$ by $\sqrt{6}$.
21. $\sqrt[4]{\frac{2}{3}}$ by $\sqrt[3]{\frac{1}{2}}$.
22. $a\sqrt{x}$ by $b\sqrt{x}$.
23. $\sqrt{2a}$ by $\sqrt[3]{4a^2}$.
24. $3\sqrt{x}-5$ by $2\sqrt{x}$.

Perform the indicated multiplications:

25. $(6\sqrt{2}-4\sqrt{6})^2$.
26. $(2-\sqrt{3})^3$.
27. $(4-2\sqrt{3})^2$.
28. $(\sqrt{6}-\sqrt{2})^2$.
29. $(\sqrt{7}+\sqrt{8})^2$.
30. $(\sqrt{a}-\sqrt{b})^2$.
31. $(\sqrt{a}-\sqrt{b})^3$.
32. $(3\sqrt{2}-2\sqrt{3})^2$.
33. $(\sqrt[3]{1}-\sqrt[3]{2})^3$.
34. $(\sqrt{a}+\sqrt{b})\times\sqrt{ab}$.
35. $(1+\sqrt{x}+1)(1-\sqrt{x+1})$.
36. $(\sqrt[3]{9}-2\sqrt[3]{4})(4\sqrt[3]{3}+\sqrt[3]{2})$.
37. $(3\sqrt{a^2+1}+4a)(2\sqrt{a^2+1}-a)$.
38. $(4\sqrt{a}+\sqrt{b})(\sqrt{a}-4\sqrt{b})$.
39. $\sqrt[3]{7}\times\sqrt{2}\times\sqrt[6]{5}$.
40. $3\sqrt{2}\times 4\sqrt[3]{3}\times 5\sqrt[4]{5}$.
41. $\sqrt[3]{12}\times\sqrt[2]{\frac{1}{8}}\times\sqrt[4]{\frac{1}{6}}$.
42. $(\sqrt{5}+\sqrt{3})(\sqrt{5}-\sqrt{3})$.
43. $(4\sqrt{5}-2\sqrt{3})(4\sqrt{5}+2\sqrt{3})$.
44. $(\sqrt{x+1}+\sqrt{x-1})(\sqrt{x+1}-\sqrt{x-1})$.
45. $(2\sqrt{6}+5\sqrt{3}-7\sqrt{2})(\sqrt{6}-2\sqrt{3}+4\sqrt{2})$.

DIVISION OF SURDS

314. Since $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} \equiv \sqrt[n]{\frac{a}{b}}$ (Art. 269), it follows that the quotient of two surds may be found *by first reducing the surds of different orders to the same order, and then taking the quotient of the rational factors for one factor of the quotient and the quotient of the factors under the radical sign for its other factor, reducing the result to its simplest form.*

1. Divide $\sqrt{18}$ by $\sqrt{6}$.

$$\frac{\sqrt{18}}{\sqrt{6}} = \sqrt{\frac{18}{6}} = \sqrt{3}.$$

2. Divide $\sqrt[3]{3}$ by $\sqrt{2}$.

$$\frac{3^{\frac{1}{3}}}{2^{\frac{1}{2}}} = \frac{3^{\frac{2}{6}}}{2^{\frac{3}{6}}} = \frac{\sqrt[6]{3^2}}{\sqrt[6]{2^3}}$$

$$= \sqrt[6]{\frac{2^3 \times 3^2}{2^6}}$$

$$= \tfrac{1}{2}\sqrt[6]{8 \times 9} = \tfrac{1}{2}\sqrt[6]{72}.$$

3. Divide $\sqrt{5} + \sqrt{3}$ by $\sqrt{5} - \sqrt{3}$.

Since $(a + b)(a - b) \equiv a^2 - b^2$, it follows that

$$(\sqrt{5} + \sqrt{3})(\sqrt{5} - \sqrt{3}) = (\sqrt{5})^2 - (\sqrt{3})^2 = 5 - 3, \text{ or } 2.$$

Hence it is seen that if the dividend and divisor be written as a fraction and both be multiplied by $\sqrt{5} + \sqrt{3}$, the denominator becomes a *rational* expression.

That is,

$$\frac{(\sqrt{5} + \sqrt{3})(\sqrt{5} + \sqrt{3})}{(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})} = \frac{5 + 2\sqrt{15} + 3}{5 - 3} = \frac{8 + 2\sqrt{15}}{2} = 4 + \sqrt{15}.$$

This is the best way to divide one quadratic surd by another. The multiplier used to render the denominator rational is called the *conjugate* of the denominator. It consists of the terms of the given denominator, connected by a different sign.

Divide:

4. $\sqrt{9}$ by $\sqrt{3}$.
5. $\sqrt{5}$ by $\sqrt{3}$.
6. $\sqrt{10}$ by $2\sqrt{5}$.
7. $\sqrt{3}$ by $\sqrt[3]{2}$.
8. $2\sqrt{5}$ by $3\sqrt{15}$.
9. $\sqrt[3]{81}$ by $\sqrt[3]{6}$.
10. $6\sqrt[3]{3}$ by $2\sqrt{2}$.
11. $\sqrt{10}$ by $\sqrt[6]{40}$.
12. $\sqrt[3]{4}$ by $\sqrt[4]{2}$.
13. 2 by $\sqrt[3]{a}$.
14. $\sqrt[3]{49}$ by $\sqrt{7}$.
15. $\sqrt[3]{3a}$ by $\sqrt[2]{7a^3}$.
16. $\sqrt[3]{a^2}$ by $\sqrt[4]{b}$.
17. $8a\sqrt{6a}$ by $4\sqrt[3]{2a^2}$.
18. $\sqrt[3]{6a^2b}$ by $\sqrt[5]{2a^3b^2}$.
19. $\sqrt[3]{15a^2b}$ by $\sqrt[4]{ab}$.

Perform the divisions indicated, by first making the denominators rational:

20. $(\sqrt{15}-\sqrt{3}) \div \sqrt{3}$.
21. $(\sqrt{2}+\sqrt{3}) \div (\sqrt{2}-\sqrt{3})$.
22. $(\sqrt{8}-\sqrt{6}) \div (\sqrt{12}-\sqrt{3})$.
23. $(3\sqrt{2}-1) \div (3\sqrt{2}+1)$.
24. $(2\sqrt{7}-3\sqrt{5}) \div (3\sqrt{7}+2\sqrt{5})$.
25. $(3\sqrt{2}+2\sqrt{3}) \div (5\sqrt{2}+4\sqrt{3})$.
26. $(2\sqrt{3}+7\sqrt{2}) \div (5\sqrt{3}-4\sqrt{2})$.
27. $(2\sqrt{3}+3\sqrt{2}) \div (2\sqrt{3}+4\sqrt{2})$.
28. $(2\sqrt{7}+3\sqrt{2}) \div (9+2\sqrt{14})$.
29. $(5\sqrt{3}-3\sqrt{5}) \div (\sqrt{5}-\sqrt{3})$.

315. Rationalization of Surds.

The process of rendering a surd or surd expression rational is called **Rationalization**.

316. When the product of two surds or surd expressions is rational, each is called a **Rationalizing Factor** of the other.

Thus, since $a^{\frac{4}{5}} \times a^{\frac{1}{5}} \equiv a$, $a^{\frac{1}{5}}$ is the rationalizing factor of $a^{\frac{4}{5}}$; and since $5\sqrt{2}-\sqrt{3}$ multiplied by $5\sqrt{2}+\sqrt{3}$ equals $(5\sqrt{2})^2-(\sqrt{3})^2 = 50-3$, or 47, $5\sqrt{2}-\sqrt{3}$ is the rationalizing factor of $5\sqrt{2}+\sqrt{3}$.

317. When two binomial quadratic surds differ only in the sign which connects their terms, they are called **Conjugate Surds.**

Thus, $a+\sqrt{b}$ and $a-\sqrt{b}$ are conjugate surds.

318. *The conjugate of a binomial quadratic surd is a rationalizing factor of that surd.*

Thus, the product of $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$ is $a-b$, a rational expression.

1. Find the simplest rationalizing factor for $\sqrt{5x}$.

Since $\sqrt{5x}\times\sqrt{5x}=\sqrt{(5x)^2}=5x$, $\sqrt{5x}$ is the factor required.

2. Find the simplest rationalizing factor for $\sqrt[3]{4}$.

$$4^{\frac{1}{3}}\times 4^{\frac{2}{3}}=4.$$

$\therefore$ $4^{\frac{2}{3}}$, or $\sqrt[3]{4^2}$, is the factor required.

Observe that a monomial with a proper fraction for an exponent is rationalized by multiplying it by itself with an exponent which, added to the given fractional exponent, makes the latter integral.

3. Find the rationalizing factor for $\sqrt{7}-\sqrt{3}$.

By Art. 118, III, $(\sqrt{7}-\sqrt{3})(\sqrt{7}+\sqrt{3})=(\sqrt{7})^2-(\sqrt{3})^2=7-3$, or 4.

$\therefore$ $\sqrt{7}+\sqrt{3}$, the conjugate of $\sqrt{7}-\sqrt{3}$, is the required factor.

4. Find the approximate value of $\dfrac{5}{\sqrt{3}}$.

Multiplying both numerator and denominator of $\dfrac{5}{\sqrt{3}}$ by $\sqrt{3}$, we have

$$\frac{5}{\sqrt{3}}=\frac{5\sqrt{3}}{\sqrt{3}\times\sqrt{3}}=\frac{5\sqrt{3}}{3}=\tfrac{5}{3}\text{ of }\sqrt{3}=\tfrac{5}{3}\text{ of }1.732.$$

It is evidently a simpler operation numerically to find $\frac{5}{3}$ of 1.732 than to divide 5 by the three place decimal 1.732, as would be necessary without rationalizing the denominator.

Find the simplest rationalizing factor for each of the following expressions:

5. $\sqrt{3}$.
6. $\sqrt[4]{ab^3}$.
7. $\sqrt{mn}$.
8. $\sqrt[3]{2}$.
9. $\sqrt[n]{m}$.
10. $\sqrt[5]{a^4}$.
11. $a^{\frac{1}{5}}b^{\frac{3}{4}}c^{\frac{1}{8}}$.
12. $\sqrt[7]{m^5n^4}$.

Express with rational denominators:

13. $\frac{2}{\sqrt{3}}$.

14. $\frac{2}{\sqrt[3]{5}}$.

15. $\frac{1}{\sqrt{a}}$.

16. $\frac{a}{\sqrt{x}}$.

17. $\frac{\sqrt{x}}{\sqrt{y}}$.

18. $\frac{n}{b\sqrt{m}}$.

19. $\frac{3}{2\sqrt[3]{2}}$.

20. $\frac{4a}{\sqrt[3]{3a^2}}$.

21. $\frac{x\sqrt{3}}{\sqrt{3}x}$.

22. $\frac{1}{\sqrt[4]{8}}$.

23. $\frac{6\sqrt{6a}}{7\sqrt{7a}}$.

24. $\frac{48}{5\sqrt{32}}$.

25. $\frac{3}{\sqrt{5}-\sqrt{2}}$.

26. $\frac{8}{\sqrt{7}+\sqrt{3}}$.

27. $\frac{a}{b+\sqrt{c}}$.

28. $\frac{1}{\sqrt{x}-\sqrt{y}}$.

29. $\frac{2-\sqrt{5}}{2+\sqrt{5}}$.

30. $\frac{\sqrt{3}-\sqrt{2}}{\sqrt{3}+\sqrt{2}}$.

31. $\frac{a^2-b^2}{\sqrt{a-b}}$.

32. $\frac{1}{2\sqrt{2}-\sqrt{3}}$.

33. $\frac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$.

34. $\frac{5}{\sqrt{1+x^2}-x}$.

35. $\frac{8+4\sqrt{3}}{6-2\sqrt{3}}$.

36. $\frac{\sqrt{x+1}+3}{\sqrt{x+1}+2}$.

37. $\frac{\sqrt{x-2}+\sqrt{x}}{\sqrt{x-2}-\sqrt{x}}$.

38. $\frac{3\sqrt{6}+\sqrt{3}}{2\sqrt{6}-\sqrt{3}}$.

39. $\frac{m\sqrt{n}+n\sqrt{m}}{\sqrt{m}+\sqrt{n}}$.

40. $\frac{x-\sqrt{x^2-y^2}}{x+\sqrt{x^2-y^2}}$.

41. $\frac{\sqrt{2}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

42. $\frac{\sqrt{a+b}-\sqrt{a-b}}{\sqrt{a+b}+\sqrt{a-b}}$.

Find the approximate values to four decimal places:

43. $\frac{3}{\sqrt{2}}$.

44. $\frac{2}{\sqrt{3}}$.

45. $\frac{2}{2-\sqrt{2}}$.

46. $\frac{\sqrt{3}}{\sqrt{2}}$.

47. $\frac{1+\sqrt{3}}{1-\sqrt{3}}$.

48. $\frac{1+\sqrt{2}}{2-\sqrt{2}}$.

49. $\frac{1}{\sqrt{2}+\sqrt{3}}$.

50. $\frac{2}{\sqrt{5}-\sqrt{3}}$.

51. $\frac{\sqrt{2}}{5+\sqrt{2}}$.

52. $\frac{-2}{\sqrt{3}-2}$.

53. $\frac{2-\sqrt{5}}{2+\sqrt{5}}$.

54. $\frac{22}{7-\sqrt{5}}$.

319. Square Roots of Binomial Surds.

Since $[\pm(a+b)]^2 \equiv a^2 + 2ab + b^2$, it follows that if any surd expression can be brought to the form $a^2 + 2ab + b^2$, the roots will be in the form $\pm(a+b)$, and may usually be found by inspection.

1. Find the square root of $8 + 2\sqrt{15}$.

To write $8 + 2\sqrt{15}$ in the required form, we must find two numbers whose sum is 8 and whose product is 15.

Since $3 + 5 = 8$, and $3 \times 5 = 15$,

$$8 + 2\sqrt{15} = 5 + 2\sqrt{5 \times 3} + 3$$
$$= [\pm(\sqrt{5} + \sqrt{3})]^2.$$

$\therefore$ the required root is $\pm(\sqrt{5} + \sqrt{3})$.

2. Find the square root of $16 + 6\sqrt{7}$.

We first attempt to make the second term correspond to $2ab$ in the type form. Hence we reduce $16 + 6\sqrt{7}$ to the form $16 + 2 \times 3\sqrt{7}$, or $16 + 2\sqrt{7} \times 9$.

And since $7 + 9 = 16$, and $7 \times 9 = 63$,

$$16 + 6\sqrt{7} = 7 + 2\sqrt{63} + 9$$
$$= [\pm(\sqrt{7} + \sqrt{9})]^2.$$

$\therefore$ the required root is $\pm(\sqrt{7} + \sqrt{9})$, or $\pm(3 + \sqrt{7})$.

Find the square roots of each of the following expressions:

3. $3 + 2\sqrt{2}$.
4. $5 - 2\sqrt{6}$.
5. $7 + 4\sqrt{3}$.
6. $18 + 8\sqrt{5}$.
7. $11 - 6\sqrt{2}$.
8. $7 - 2\sqrt{10}$.
9. $5 + \sqrt{24}$.
10. $6 + \sqrt{11}$.
11. $27 - 10\sqrt{2}$.
12. $a^2 + 2a\sqrt{b} + b$.
13. $a + 4\sqrt{ab} + 4b$.
14. $a - 2\sqrt{a-1}$.

RADICAL EQUATIONS

320. An equation containing an irrational root of an unknown number in either or both of its members is called an **Irrational**, or **Radical Equation**.

Thus, $\sqrt{x} + 5 = 8$, and $\sqrt[3]{x^2} - 9 = 3$ are radical equations.

321. Many radical equations can be reduced to equivalent simple equations and solved as such by first clearing them of radicals, or *rationalizing* them. Others lead to equations of higher degree than the first, and are treated later.

322. Quadratic radicals may be rationalized by transposing all the terms to the first member and multiplying both members by the conjugate of the first.

But the best way to clear of radicals is to transfer the terms so that one radical (preferably the most complicated one) shall be one member, and then raise both members to the same power, repeating the operation if necessary.

If the equation contains fractions, it should be cleared of them first of all.

323. Since squaring an equation is equivalent to multiplying both members by a factor containing the unknown number, it is *possible* that extraneous roots may be introduced in the process of reduction to equivalent simple equations (Art. 204, 1). However, in solving *radical* equations, we must not expect to find new roots introduced in every case where the theory for *integral* equations would give rise to them.

Thus, by squaring, $x = 2$ becomes $x^2 = 4$, an equation containing the root $x = 2$, and also the extraneous root $x = -2$. But if we square the equation $\sqrt{x} = 2$, we get the equivalent simple equation $x = 4$, and this value satisfies the original equation, no extraneous root being introduced.

Again, by squaring the equation $\sqrt{x} = -2$, $x = 4$; substituting this value for x, $\sqrt{4} = -2$, which satisfies the equation if $\sqrt{x}$ is regarded as both $+2$ and -2, but is extraneous if only the *positive* sign is taken with the radical. This restriction makes the equation insolvable. If the equation $-\sqrt{x} = -2$ is squared, we get $x = 4$, which satisfies the given equation.

324. Whether certain roots are to be considered extraneous or not depends upon the values which are used in verifying radical equations. For, since $a^2 + 2ab + b^2$ is evidently the square of $\pm(a + b)$, the square root of $a^2 + 2ab + b^2$ is *both*

plus and minus, and hence $\sqrt{a^2 + 2ab + b^2}$ strictly represents both the positive and negative square roots.

It is customary, however, to consider *only the positive roots* of the radical (unless the contrary is stated), *and to reject as extraneous all roots which do not satisfy the original equation.*

The effect of admitting or rejecting negative values is shown in the illustrative solutions.

1. Solve $\sqrt{x+1} + \sqrt{x-2} = 3$.

Transposing, $\sqrt{x+1} - 3 = -\sqrt{x-2}$.

Squaring, $x + 1 - 6\sqrt{x+1} + 9 = x - 2$,

or $\sqrt{x+1} = 2$.

Squaring, $x + 1 = 4$.

$\therefore \quad x = 3$.

Observe that this is equivalent to the following solution:

By transposing, the given equation becomes

$$\sqrt{x+1} + \sqrt{x-2} - 3 = 0. \tag{1}$$

Multiplying by the rationalizing factor, $\sqrt{x+1} - \sqrt{x-2} - 3$,

we have $(\sqrt{x+1} + \sqrt{x-2} - 3)(\sqrt{x+1} - \sqrt{x-2} - 3) = 0$, (2)

or $x + 1 - 6\sqrt{x+1} - (x-2) + 9 = 0$,

or $\sqrt{x+1} - 2 = 0$, (3)

which is the result obtained by squaring the first time.

Next, if we take the product of the other two of the four factors, obtained by changing the signs of the radicals in all possible ways, viz.,

$$(-\sqrt{x+1} - \sqrt{x-2} - 3)(-\sqrt{x+1} + \sqrt{x-2} - 3) = 0, \tag{4}$$

we get $x + 1 + 6\sqrt{x+1} - (x-2) + 9 = 0$,

or $\sqrt{x+1} + 2 = 0$, (5)

which is the conjugate of (3).

Then $(\sqrt{x+1} - 2)(\sqrt{x+1} + 2) = 0$,

or $x + 1 - 4 = 0$.

$\therefore \quad x = 3$.

From the above it is seen that the simple equation $x - 3 = 0$ is really the product of the four equations

$$\sqrt{x+1} + \sqrt{x-2} - 3 = 0 \quad (A)$$
$$\sqrt{x+1} - \sqrt{x-2} - 3 = 0 \quad (B)$$
$$-\sqrt{x+1} + \sqrt{x-2} - 3 = 0 \quad (C)$$
$$-\sqrt{x+1} - \sqrt{x-2} - 3 = 0 \quad (D)$$

formed by admitting both *positive* and *negative* signs with the radicals, and hence contains all their roots. It follows that any root which satisfies any one of the four equations is in this broader sense a root of the given equation.

2. Solve the equation $\sqrt{x+1} - \sqrt{x-2} = 3$.

This is equation (B) in the preceding solution, and it is readily seen that the same rational equation, $x - 3 = 0$, results equally from (A), (B), (C), or (D), since each is rationalized by finding the product of all of them. The same result is obtained by the method of squaring.

Thus, $$\sqrt{x+1} - 3 = \sqrt{x-2}.$$

Squaring, $$x + 1 - 6\sqrt{x+1} + 9 = x - 2,$$

or $$\sqrt{x+1} = 2.$$

Squaring, $$x + 1 = 4.$$

$$\therefore x = 3, \text{ as in Ex. 1.}$$

CHECK. $\sqrt{4} - \sqrt{1} = 3$, or $(\pm 2) - (\pm 1) = 3$.

This is true in $(+2) - (-1) = 3$, but it is not true if we consider only the positive root of the radicals. For in the more restricted sense of that custom the root 3 is usually called extraneous, and the equation is said to be *impossible*.

3. Solve the equation $\sqrt{x+2} + \sqrt{x-6} = 2\sqrt{x-3}$.

Squaring the given equation,

$$x + 2 + 2\sqrt{x^2 - 4x - 12} + x - 6 = 4x - 12,$$

or $$2\sqrt{x^2 - 4x - 12} = 2x - 8,$$

or $$\sqrt{x^2 - 4x - 12} = x - 4.$$

Squaring, $$x^2 - 4x - 12 = x^2 - 8x + 16,$$

or $$4x = 28.$$

$$\therefore x = 7.$$

CHECK. $$\sqrt{7+2} + \sqrt{7-6} = 2\sqrt{7-3},$$
$$\sqrt{9} + \sqrt{1} = 2\sqrt{4},$$
$$3 + 1 = 2 \times 2.$$

If the equation contains three or more radicals, no general direction for solving can be given. To facilitate the work, the student must exercise his best judgment in transposing terms before squaring. The special nature of each equation must be his guide.

Solve the following radical equations, and by checking determine the roots commonly called extraneous:

4. $\sqrt{x+2}=5.$
5. $\sqrt{x-6}=5.$
6. $\sqrt{2x+6}=4.$
7. $\sqrt[3]{2x-3}=-3.$
8. $\sqrt{x^2-3}-x=-2.$
9. $\sqrt{x}+\sqrt{x+6}=3.$
10. $\sqrt{4+\sqrt{x-2}}=3.$
11. $\sqrt{x+5}+\sqrt{x}=5.$
12. $\sqrt{45+x}+\sqrt{x}=9.$
13. $\sqrt{x+6}+\sqrt{x+3}=3.$
14. $\dfrac{\sqrt{x}-3}{\sqrt{x}-6}=\dfrac{\sqrt{x}-1}{\sqrt{x}-5}.$
15. $\sqrt{x}=3.$
16. $\sqrt{3x+1}=4.$
17. $\sqrt{x^2+7}-1=x.$
18. $\sqrt{x^2-7}=7-x.$
19. $\sqrt{x}-\sqrt{x+9}=-1.$
20. $\sqrt{x-32}=16-\sqrt{x}.$
21. $\sqrt{9x+8}+\sqrt{9x}=4.$
22. $\sqrt{5x-1}=2\sqrt{x+3}.$
23. $\sqrt{x+2}-\sqrt{x-2}=\sqrt{2}.$
24. $\sqrt{x-2}+\sqrt{x+2}=\sqrt{2}.$
25. $\dfrac{\sqrt{x-2}}{\sqrt{x-4}}=\dfrac{\sqrt{x+1}}{\sqrt{x-3}}.$
26. $\dfrac{1}{\sqrt{x}}-\sqrt{x+2}=\sqrt{x}.$
27. $\sqrt{2x-1}+\sqrt{2x+4}=5.$
28. $2\sqrt{x}-\sqrt{4x-22}=\sqrt{2}.$
29. $\sqrt{x+5}+\sqrt{x-3}=2\sqrt{x}.$
30. $\sqrt{8x+17}-\sqrt{2x}-\sqrt{2x+9}=0.$
31. $\sqrt{x+9}-\sqrt{x-5}=\dfrac{7}{\sqrt{x+5}}.$
32. $2+\sqrt{4x^2-9x+8}-2x=0.$
33. $1+\sqrt{2x}=\sqrt{2x+\sqrt{3+8x}}.$

IMAGINARY AND COMPLEX NUMBERS

325. By the introduction of negative, fractional, and surd numbers we have been enabled to solve many equations that otherwise would have been without solution, but rational numbers and surds do not meet all the demands upon a number system. For instance, if we attempt to solve the equation $x^2 = -1$, we must say that the solution is impossible or we must extend the idea of number to include the new symbol $\pm\sqrt{-1}$ to which the equation leads.

326. It is evident that $\sqrt{-1}$ is neither $+1$ nor -1, since no number thus far considered gives *a negative result when squared.* But the even roots of negative numbers give rise to results of the form $\pm\sqrt{-1}$, or $\pm\sqrt{-a}$, and hence, if our number system is to be perfectly general, there is need for a new form of number whose square shall be negative, that is to say, a number whose meaning shall be deduced from the relation $(\sqrt{-1})^2 = -1$, and which, subject to this relation, shall be such that we can operate with it by means of the ordinary laws of algebra.

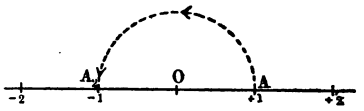
327. An even root of a negative number is called an **Imaginary Number.**

Thus, $\sqrt{-1}$, $\sqrt[4]{-2}$, $\sqrt{3}\sqrt{-2}$, $\sqrt{-a}$ are imaginary numbers. In distinction from imaginaries, all the numbers hitherto described are called *real numbers.*

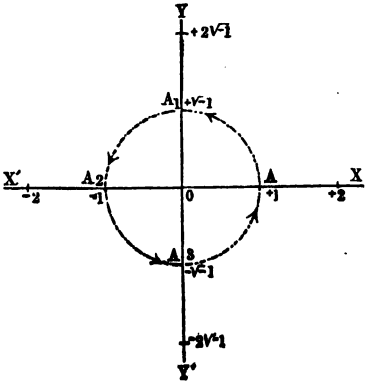
The designation "imaginary" for this new form of number is somewhat misleading, since these numbers are essentially no less real than are the rational fraction and negative numbers.

As we have seen, fractional and negative results may or may not admit of a concrete interpretation in particular problems; imaginaries can have no interpretation in strictly arithmetical problems, as they are not expressible as rational numbers or approximately in terms of such numbers. Fractional, negative, and surd numbers were introduced to enable us to give a formal solution of many problems otherwise impossible. If we extend the system of real numbers by the introduction of imaginaries, we shall be enabled to perform all algebraic operations, including the solution of equations, without the limitations otherwise necessary.

328. We have already seen that positive and negative numbers are conveniently represented as standing on opposite sides of zero on a line laid off into units of length, as in the figure. And since $+1 \times -1 = -1$, we may look upon the multiplication of $+1$ by -1 as turning the line $0A$ about 0 through 180° to the position $0A_2$, in the direction indicated by the arrows (counter-clockwise); that is, as swinging the line $0A$ from the position in which it represents $+1$ to the position in which it represents -1.



329. By a similar method of graphic representation, we are enabled to give a clearer idea of the nature of imaginary numbers, and their relation to real numbers. For, since $(\sqrt{-1})^2 = \sqrt{-1} \times \sqrt{-1} = -1$, by definition, it follows that if multiplication by -1 (that is, by $\sqrt{-1} \times \sqrt{-1}$), turns $0A$ through 180°, it is natural to say that multiplication by $\sqrt{-1}$ should turn it through 90°, to the position $0A_1$. Since



$0A$ represents the positive unit $+1$, this may be shown as follows:

$$0A_1 = 1 \cdot \sqrt{-1} \quad = +\sqrt{-1},$$
$$0A_2 = 1 \cdot (\sqrt{-1})^2 = -1,$$
$$0A_3 = 1 \cdot (\sqrt{-1})^3 = -\sqrt{-1},$$
$$0A \quad = 1 \cdot (\sqrt{-1})^4 = +1.$$

Hence, if we draw the axes XX', YY', intersecting at right angles at 0 (Art. 244), as in the annexed figure, and regard the symbols $+\sqrt{-1}$ and $-\sqrt{-1}$ as denoting *direction* as well as *length* (just as $+$ and $-$ as signs of *quality* may indicate direction), we may represent $+1\sqrt{-1}$ (or $+\sqrt{-1}$), $+2\sqrt{-1}$, $+3\sqrt{-1}$, etc., by equal divisions laid off on the perpendicular $0Y$, upward from 0, and $-1\sqrt{-1}$, $-2\sqrt{-1}$, $-3\sqrt{-1}$, etc., by equal divisions laid off on the same line, downward from 0.

That is, just as $+2$ indicates 2 units to the right of 0 and -2 indicates 2 units to the left, so $+2\sqrt{-1}$ indicates 2 units up from 0 and $-2\sqrt{-1}$ indicates 2 units down.

330. In the above graphic representation of the relation between real and imaginary numbers, XX' is called the *axis of real numbers*, and YY' the *axis of imaginary numbers*. It is to be observed that if YY' were taken as the axis of real numbers, then XX' would become the axis of imaginaries. In this representation it is seen that imaginaries have as much reality as do real numbers.

331. In introducing these new numbers into the number system of algebra, we are tacitly assuming that, except so far as the relation $(\sqrt{-1})^2 = -1$, or in general $(\sqrt[2n]{-a})^{2n} = -a$, may determine the quality of the result, they obey all the laws established for other numbers, and as a consequence that they may be used either alone or in combination with real numbers in all algebraic operations.

332. Reduction to the Form $a\sqrt{-1}$.

Since $\sqrt{ab} = \sqrt{a} \times \sqrt{b}$, it follows that $\sqrt{-2}$ may be written $\sqrt{2 \times -1}$, or $\sqrt{2} \times \sqrt{-1}$. Hence any number of the form $\sqrt{-c}$ may be written in the form $a\sqrt{-1}$, where a is a real number and $\sqrt{-1}$ the imaginary part, "imaginary unit" as it is called.

Thus, $\sqrt{-3} = \sqrt{3 \times -1} = \sqrt{3}\sqrt{-1}$; $\sqrt{-4} = \sqrt{4 \times -1} = \sqrt{4} \times \sqrt{-1}$, or $\pm 2\sqrt{-1}$, where only the positive result is used unless otherwise stated.

Represent graphically the following imaginary numbers:

1. $3\sqrt{-1}$.
2. $4\sqrt{-1}$.
3. $-2\sqrt{-1}$.
4. $+\frac{1}{2}\sqrt{-1}$.
5. $\sqrt{3}\sqrt{-1}$.
6. $2\sqrt{2}\sqrt{-1}$.

Reduce the following expressions to the form $a\sqrt{-1}$:

7. $\sqrt{-2}$.
8. $\sqrt{-9}$.
9. $\sqrt{-28}$.
10. $2\sqrt{-4}$.
11. $-\sqrt{-24}$.
12. $\sqrt{-a^3}$.
13. $-2\sqrt{-4}$.
14. $\sqrt{3}\sqrt{-3}$.
15. $\sqrt{-(m-n)^2}$.

333. The algebraic sum of a real number and an imaginary number is called a **Complex Number.**

Thus, $3 + \sqrt{-1}$, $2 - \sqrt{-1}$, $a \pm b\sqrt{-1}$ are complex numbers.

334. For brevity the symbol $\sqrt{-1}$ is often written i.

Thus, $\sqrt{-1} = i$, $2\sqrt{-1} = 2i$, $\sqrt{-5} = i\sqrt{5}$, etc.; also $(\sqrt{-1})^2 = (\pm i)^2 = -1$.

335. In addition to the double series of real numbers

$$\cdots -3, -2, -1, 0, +1, +2, +3, \cdots,$$

we now also have the double series of imaginary numbers

$$\cdots -3i, -2i, -i, 0, +i, +2i, +3i, \cdots.$$

Obviously these series have no number in common except 0. Verify this by reference to the graphic representation given in Art. 329.

ADDITION AND SUBTRACTION OF IMAGINARIES

336. In the addition and subtraction of imaginary and complex numbers, the laws established for other irrational numbers apply. Hence, to add such numbers, reduce them to the form $a\sqrt{-1}$, or $a+b\sqrt{-1}$ if complex, and then add the real parts and the imaginary parts separately, writing the results as an algebraic sum.

A similar method applies in subtraction.

1. Find the sum of $\sqrt{-4}$, $\sqrt{-9}$, and $\sqrt{-3}$.

$$\sqrt{-4} = \sqrt{4}\times\sqrt{-1} = 2\sqrt{-1}$$
$$\sqrt{-9} = \sqrt{9}\times\sqrt{-1} = 3\sqrt{-1}$$
$$\sqrt{-3} = \sqrt{3}\times\sqrt{-1} = \sqrt{3}\sqrt{-1}$$
$$\therefore \text{ the sum} = (5+\sqrt{3})\sqrt{-1}$$

2. Simplify $\sqrt{-9}+\sqrt{-16}-\sqrt{-9}$.

$$\sqrt{-9} = 3\sqrt{-1}$$
$$\sqrt{-16} = 4\sqrt{-1}$$
$$\sqrt{-9} = 3\sqrt{-1}$$
$$\therefore \sqrt{-9}+\sqrt{-16}-\sqrt{-9} = 3\sqrt{-1}+4\sqrt{-1}-3\sqrt{-1} = 4\sqrt{-1}.$$

3. Add $2-2\sqrt{-1}$, $5+\sqrt{-3}$, and $3+\sqrt{-16}$.

$$2-2\sqrt{-1} = 2- \qquad 2\sqrt{-1} = 2-2i$$
$$5+\sqrt{-3} = 5+ \qquad \sqrt{3}\sqrt{-1} = 5+i\sqrt{3}$$
$$3+\sqrt{-16} = 3+ \qquad 4\sqrt{-1} = 3+4i$$
$$\therefore 10+(2+\sqrt{3})\sqrt{-1} = 10+i(2+\sqrt{3}).$$

Simplify each of the following expressions:

4. $2\sqrt{-12}+3\sqrt{-27}$.

5. $7\sqrt{-81}+5\sqrt{-144}$.

6. $\sqrt{-27}+\sqrt{-48}+\sqrt{-75}$.

7. $\sqrt{-a^2}+\sqrt{-9a^2}$.

8. $\sqrt{-4}+\sqrt{-5}+\sqrt{-16}$.

9. $(3+\sqrt{-4})+(5-2\sqrt{-9})$.

10. $(3-2\sqrt{-5})-(2-3\sqrt{-5})$.

11. $\sqrt{-9a^2}+\sqrt{-4b^2}-\sqrt{-7c^2}$.

12. $2\sqrt{-4}+3\sqrt{-25}-6\sqrt{-9}$.

13. $\sqrt{-1}+\sqrt{-2}+\sqrt{-3}+\sqrt{-4}$.

14. $\sqrt{-\frac{1}{9}}-\sqrt{-\frac{4}{81}}+\sqrt{-\frac{1}{4}}$.

15. $\sqrt{-a^6}+3\sqrt{-16a^6}-2\sqrt{-9a^6}$.

MULTIPLICATION OF IMAGINARIES

337. To find the product when imaginary or complex numbers are involved, first reduce them to the form $a\sqrt{-1}$, or $a+b\sqrt{-1}$, and then multiply as in other radicals, but observing that $\sqrt{-1}\times\sqrt{-1}=-1$.

1. Find the product of $\sqrt{-5}$ by $\sqrt{-3}$.

$$\sqrt{-5}=\sqrt{5}\,\sqrt{-1}.$$
$$\sqrt{-3}=\sqrt{3}\,\sqrt{-1}.$$
$$\therefore\ \sqrt{-5}\times\sqrt{-3}=(\sqrt{5}\,\sqrt{-1})(\sqrt{3}\,\sqrt{-1})$$
$$=\sqrt{15}\,(\sqrt{-1})^2$$
$$=-\sqrt{15},\ \text{since}\ (\sqrt{-1})^2=-1.$$

2. Multiply $3+\sqrt{-2}$ by $2-\sqrt{-5}$.

$$\begin{array}{rl} 3+\sqrt{-2} & =3+\sqrt{2}\,\sqrt{-1} \\ 2-\sqrt{-5} & =2-\sqrt{5}\,\sqrt{-1} \\ \hline & 6+2\sqrt{2}\,\sqrt{-1} \\ & \quad -3\sqrt{5}\,\sqrt{-1}-\sqrt{10}(-1) \\ \hline & 6+(2\sqrt{2}-3\sqrt{5})\sqrt{-1}+\sqrt{10} \end{array}$$

3. Raise $\sqrt{-1}$ to the 2d, 3d, 4th, and 5th powers.

(a)	(b)
$(\sqrt{-1})^2=-1$;	$i^2=-1$;
$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=(-1)\sqrt{-1}=-\sqrt{-1}$;	$i^3=-1\,i=-i$;
$(\sqrt{-1})^4=[(\sqrt{-1})^2]^2=(-1)^2=+1$;	$i^4=(i^2)^2=(-1)^2=+1$;
$(\sqrt{-1})^5=(\sqrt{-1})^4\sqrt{-1}=(+1)\sqrt{-1}=+\sqrt{-1}$.	$i^5=1\,i=i$.

Observe that the 5th power is the same as the first, the first four powers being respectively $+\sqrt{-1}$, -1, $-\sqrt{-1}$, and $+1$. It will be found that for higher powers these results recur in the same order. Verify by reference to the figure in Art. 329.

It is also to be observed that *any even power is real, and any odd power is imaginary.*

Find the products:

4. $4\sqrt{-27} \times 2\sqrt{-3}$.
5. $\sqrt{-2} \times \sqrt{-50}$.
6. $-\sqrt{-5} \times -3\sqrt{-10}$.
7. $5\sqrt{-2} \times 3\sqrt{-2}$.
8. $(-3\sqrt{-a})(5\sqrt{-b})$.
9. $(\sqrt{-3} - 2\sqrt{-1})^2$.
10. $(\frac{1}{2} - \frac{1}{2}\sqrt{-3})^2$.
11. $(\sqrt{-1})^6$; $(\sqrt{-1})^7$.
12. $(3+\sqrt{-5})(4-\sqrt{-3})$.
13. $(-1+\sqrt{-3})^3$.
14. $(2-3i)(2+3i)$.
15. $(2+i\sqrt{3})(2-i\sqrt{3})$.
16. $(2+\sqrt{-3})(3+\sqrt{-2})$.
17. $(3+2\sqrt{-1})(3-2\sqrt{-1})$.
18. $(\sqrt{-2}+\sqrt{-3})(\sqrt{-4}-\sqrt{-5})$.
19. $(2\sqrt{-4}-4\sqrt{-3})(5\sqrt{-4}+7\sqrt{-3})$.
20. $(\sqrt{3}-2\sqrt{-2})(\sqrt{3}+2\sqrt{-2})$.
21. $(a+b\sqrt{-1})(c+d\sqrt{-1})$.
22. $(\sqrt{-2}+3\sqrt{-3})(4\sqrt{-2}-\sqrt{-3})$.
23. $(2\sqrt{-3}+3\sqrt{-6})(3\sqrt{-3}-5\sqrt{-2})$.

DIVISION OF IMAGINARIES

338. The quotients of imaginary or complex numbers may be found by first reducing them to the form $a\sqrt{-1}$, or $a+b\sqrt{-1}$, as the case may be, and then dividing as in other radicals, observing the principles used in multiplication.

In practice it is best to write the dividend and divisor as a fraction, then rationalize the denominator, and simplify the result.

1. Divide $\sqrt{-15}$ by $\sqrt{-3}$.

$$\frac{\sqrt{-15}}{\sqrt{-3}} = \frac{\sqrt{15}\sqrt{-1}}{\sqrt{3}\sqrt{-1}} = \frac{\sqrt{15}}{\sqrt{3}} = \sqrt{5},$$

2. Divide $\sqrt{27}$ by $\sqrt{-3}$.

$$\frac{\sqrt{27}}{\sqrt{-3}} = \frac{3\sqrt{3}}{\sqrt{3}\sqrt{-1}} = \frac{3}{\sqrt{-1}}.$$

But $$\frac{3}{\sqrt{-1}} = \frac{3 \times \sqrt{-1}}{\sqrt{-1} \times \sqrt{-1}} = \frac{3\sqrt{-1}}{-1} = -3\sqrt{-1}.$$

3. Divide $5 - \sqrt{-2}$ by $1 + \sqrt{-2}$.

$$\frac{5-\sqrt{-2}}{1+\sqrt{-2}} = \frac{(5-\sqrt{2}\sqrt{-1})(1-\sqrt{2}\sqrt{-1})}{(1+\sqrt{2}\sqrt{-1})(1-\sqrt{2}\sqrt{-1})}$$

$$= \frac{5-6\sqrt{2}\sqrt{-1}+2(-1)}{1-2(-1)}$$

$$= \frac{5-6\sqrt{2}\sqrt{-1}-2}{1+2}$$

$$= \frac{3-6\sqrt{2}\sqrt{-1}}{3}, \text{ or } 1-2\sqrt{2}\sqrt{-1}.$$

Perform the divisions indicated:

4. $\sqrt{-8} \div \sqrt{-2}$.
5. $16\sqrt{-9} \div 2\sqrt{-4}$.
6. $\sqrt{3} \div \sqrt{-3}$.
7. $12 \div -\sqrt{-2}$.
8. $\sqrt{ab} \div \sqrt{-a}$.
9. $2\sqrt{-6} \div 3\sqrt{-3}$.
10. $(\sqrt{12} + \sqrt{3}) \div \sqrt{-3}$.
11. $(1+\sqrt{-1}) \div (1-\sqrt{-1})$.
12. $(2+\sqrt{-3}) \div (2-\sqrt{-3})$.
13. $(3+2i) \div (3-2i)$.
14. $(8-\sqrt{-4}) \div (4+2\sqrt{-1})$.
15. $(3+2\sqrt{-1}) \div (2-3\sqrt{-1})$.
16. $(8+\sqrt{-1}) \div (3+2\sqrt{-1})$.
17. $(\sqrt{3}+i\sqrt{2}) \div (\sqrt{3}-i\sqrt{2})$.
18. $(a+b\sqrt{-1}) \div (a-b\sqrt{-1})$.
19. $(2\sqrt{3}-3\sqrt{-2}) \div (3\sqrt{3}+2\sqrt{-2})$.
20. $(3\sqrt{-2}+2\sqrt{-3}) \div (3\sqrt{-2}-2\sqrt{-3})$.
21. $(3\sqrt{-2}+2\sqrt{-5}) \div 3\sqrt{-2}-2\sqrt{-5}$.

QUADRATIC EQUATIONS

WITH ONE UNKNOWN NUMBER

339. An integral equation in which the highest power of the unknown number is a *square* is called a **Quadratic Equation.** It may contain only the square of the unknown number, or it may contain both the second and first powers.

Thus, $x^2 = 12$, $x^2 + 2x = 3$, and $ax^2 + bx = c$ are quadratic equations.

The term "quadratic" is derived from the Latin *quadratus*, meaning *squared*. Squaring both members of $x = 3$, we have $x^2 = 9$, or $x^2 - 9 = 0$, a quadratic equation containing only the second power of x. Squaring $x - 2 = 3$, we have $x^2 - 4x + 4 = 9$, another quadratic containing both the first and second powers of x.

340. Every quadratic equation may be made to assume the form

$$ax^2 + bx + c = 0,$$

where x is the unknown number, and a, b, and c are known numbers. The latter are called the *coefficients* of the equation. The third term c is called the *constant* term.

341. When $b = 0$, the second term, bx, is 0, and the equation contains only the second power of x; it is therefore said to be an **Incomplete** or a **Pure Quadratic.**

When b is not 0, the second term remains, and the equation contains both the first and second powers of x; it is therefore called a **Complete** or an **Affected Quadratic.**

342. When we consider an equation like $x^2 = 25$, we find that it can be solved by merely extracting the square root of both members; that is, if $x^2 = 25$, $x = \pm 5$, *two* roots.

The question naturally arises why not place the sign $\pm$ before the first member as well as before the second after the root is extracted. It is entirely proper to do so, but it is *unnecessary*, as is shown by the following:

The equation	$x^2 = 25$	
leads to	$\pm x = \pm 5,$	
and hence to	$+x = +5,$	(1)
	$+x = -5,$	(2)
	$-x = -5,$ or $+x = +5,$ (Why?)	(3)
	$-x = +5,$ or $+x = -5.$	(4)

Since we are required to solve for x, and not for $-x$, in finding the roots of the equation, it is seen that (3) and (4) reduce to (1) and (2), and hence we get the *same values* for x whether we use the sign $\pm$ in one member or in both.

It therefore appears that the use of the double sign $\pm$ in the *second member only* gives the *same roots* as its use in *both members*.

INCOMPLETE OR PURE QUADRATICS

343. Pure quadratics contain only the second power of the unknown number, hence they may be reduced to the form $x^2 = b$, from which we obtain $x = \pm\sqrt{b}$, by Art. 342.

1. Solve the equation $4x^2 - 50 = 2x^2$.

Transposing and uniting like terms,	$2x^2 = 50.$
Dividing by 2,	$x^2 = 25.$
Taking the square root,	$x = \pm 5.$

It is seen that x has two values, or, in other words, that the equation has two roots, and that they are numerically equal, but of opposite signs.

2. Solve the equation $(3x + 2)(3x - 2) = 77$.

$$9x^2 - 4 = 77, \text{ or } 9x^2 = 81,$$

whence $$x^2 = 9, \text{ and } x = \pm 3.$$

CHECK. $(9 + 2)(9 - 2) = 77$; also $(3 \times -3 + 2)(3 \times -3 - 2) = -7 \times -11 = 77$.

3. Find the roots of the equation $2x^2 = -8$.

$$2x^2 = -8.$$

$$x^2 = -4.$$

$x = \pm\sqrt{-4} = \pm\sqrt{4 \times -1} = \pm 2\sqrt{-1}$, *imaginary* roots. (Art. 327)

4. Solve the equation $\frac{x+2}{x-2}+\frac{x-2}{x+2}=3.$

$$\frac{x+2}{x-2}+\frac{x-2}{x+2}=3.$$

Clearing of fractions, $(x+2)^2+(x-2)^2=3(x+2)(x-2),$

or $x^2+4x+4+x^2-4x+4=3x^2-12.$

Transposing and uniting, $-x^2=-20.$

$\therefore x=\pm\sqrt{20}$, or $\pm 2\sqrt{5}.$

In solving fractional equations which lead to quadratics we employ the same methods used in the solution of fractional equations which lead to simple equations.

Solve the following equations:

5. $5x^2-6=2x^2+6.$
6. $2x^2+27=x^2+36.$
7. $5x^2+20=3x^2+52.$
8. $4x^2+25=7x^2-50.$
9. $(x+2)^2=4x+20.$
10. $7x^2-5=5x^2-13.$
11. $\frac{x^2-1}{3}=16.$
12. $\frac{x^2-16}{4}=12.$
13. $\frac{x^2-9}{10}=\frac{x^2+11}{15}.$
14. $\frac{8}{2x^2}-\frac{3}{4x^2}=\frac{13}{36}.$
15. $(x-4)(x+7)=3(x+10).$
16. $\frac{x-a}{x+a}+\frac{x+a}{x-a}=5.$
17. $\frac{m}{x^2-n}=\frac{n}{x^2-m}.$
18. $\frac{x+3}{8}-\frac{10}{x+1}=\frac{1}{2}.$
19. $(x+\frac{1}{2})(x-\frac{1}{2})=\frac{5}{16}.$
20. $7x^2+\frac{1}{3}=6x^2+4\frac{1}{3}.$
21. $mx^2=a^2-nx^2.$
22. $(a-x)^2=(x-a)(3x+a).$
23. $(7-x)^2-30=100-(7+x)^2.$
24. $x(x+2a)+b=a(1+2x).$
25. $\frac{3x^2}{4}+(2x^2-3)=\frac{16x^2+9}{5}.$
26. $\frac{x+5}{x-5}+\frac{x-5}{x+5}=-3\frac{3}{4}.$
27. $\frac{a}{x+b}-\frac{a}{x-b}=c.$
28. $\frac{x+2}{x-1}+\frac{x-2}{x+1}=-x^0.$
29. $(3x-4)(2x-3)=66-17x.$
30. $\frac{y}{m+n}-\frac{m-n}{y}=0.$
31. $\frac{ac}{z}-\frac{z}{a}=\frac{a}{z}.$
32. $\frac{y+b}{y-b}-\frac{2b}{1-b}=-\frac{y-b}{y+b}.$

33. $(2y-3)(3y-4)-(y-13)(y-4)=40.$

34. $\dfrac{m^2+n^2}{x^2-n^2}=\dfrac{x-m}{x-n}+\dfrac{x+m}{x+n}.$

35. $2+\dfrac{89}{(x^2+2)(x^2-3)}=\dfrac{5x^2-1}{x^2-3}-\dfrac{3x^2+1}{x^2+2}.$

36. $\dfrac{1}{x+a}+\dfrac{1}{x-b}=\dfrac{1}{x-a}+\dfrac{1}{x+b}.$

PROBLEMS

344. 1. If the square of 3 is added to the square of a certain number, the sum is the square of 5. Find the number.

2. The difference between the squares of 8 and 10 equals the square of a certain number. Find the number.

3. A field whose length is twice its width contains 20 acres. What is its length?

4. The sum of the squares of two numbers is 20, the difference 12. What are the numbers?

5. The product of a certain number increased by 7 and the same number decreased by 7 is 32. Find the number.

6. Two numbers are as 2 to 3, and the difference of their squares is 45. What are the numbers?

7. The sum of the squares of two numbers is 117, and the numbers are as 3 to 2. Find the numbers.

8. The product of 8 increased by a certain number and 8 decreased by that number is 55. What is that "certain number"?

9. A man worked a certain number of days, which was equal to his daily wages in dollars. If he got $81 for his work, how many days was he employed?

10. A certain square number diminished by the square of 6 is equal to the square of 8. What is the square number?

11. Three times the square of a certain number is 72 more than the square of the number. Find the number.

12. Five times the square of a certain number is 500 less than the square of 5 times the number. What is the number?

13. Twice the square of a number is 175 more than the square of half the number. What is the number?

14. The sum of two square numbers is 85, and their difference is 13. What are their square roots?

15. The sides of two square fields are as 2 to 3. One contains 500 square rods more than the other. What is the perimeter of the smaller field?

16. A field is twice as long as it is wide. Another field is half as long and half as wide, and contains 5 acres. How many acres in the larger field?

17. The area of the six sides of a cubical room is 1024 square feet greater than the area of the floor and ceiling combined. Find a side of the room.

18. It will require 243 tiles, each 8 inches square, to pave a hall. If square tiles of a larger size are used, 192 will be required. Find the size of the larger tile.

COMPLETE OR AFFECTED QUADRATICS

345. There are three principal methods of solving complete quadratic equations: (1) by Factoring; (2) by Completing the Square; (3) by using Formulas. These methods are all closely related, as will appear from the explanations and illustrative solutions given under each method.

346. Solution by Factoring.

It is seen from Art. 161 that when the first member of any quadratic written in the form $ax^2 + bx + c = 0$ can be resolved into two factors, each containing only the first power of x, the equation can be solved by factoring.

Thus, $x^2 - 2x - 3 = 0$ may be written $(x - 3)(x + 1) = 0$. Then $x - 3 = 0$, $x = 3$; and $x + 1 = 0$, $x = -1$.

Equations of higher degree than the second can often be solved by factoring, as will be seen in some of the solutions below.

1. Solve the equation $x^2 + 3x - 40 = 0$.

$$x^2 + 3x - 40 = (x + 8)(x - 5). \qquad \text{(Art. 150)}$$
$$\therefore (x + 8)(x - 5) = 0.$$

Since the product of the two factors is 0, either one of them can be 0. Hence,

$$x + 8 = 0, \text{ or } x - 5 = 0.$$
$$\therefore x = -8, \text{ or } x = 5.$$

CHECK. $(-8)^2 + (3 \times -8) - 40 = 64 - 24 - 40 = 0.$
$5^2 + 3 \times 5 - 40 = 25 + 15 - 40 = 0.$

2. Solve the equation $6x^2 + 7x - 3 = 0$.

$$6x^2 + 7x - 3 = (3x - 1)(2x + 3). \qquad \text{(Art. 151)}$$
$$3x - 1 = 0, \text{ or } x = \tfrac{1}{3}.$$
$$2x + 3 = 0, \text{ or } x = -\tfrac{3}{2}.$$

3. Solve the equation $x^3 + 3x^2 - 10x = 0$.

$$x^3 + 3x^2 - 10x = x(x - 2)(x + 5).$$
$$\therefore x,\ x - 2, \text{ or } x + 5 \text{ is } 0,$$

and

$$x = 0,\ 2, \text{ or } -5.$$

Solve the following by factoring:

4. $x^2 - 3x = 5x.$

5. $(x + 3)^2 = 49.$

6. $2x(x + 7) = 0.$

7. $x^2 - 3x - 10 = 0.$

8. $x^2 + 7x - 8 = 0.$

9. $x^2 + 12x + 35 = 0.$

10. $x^2 - 6x - 16 = 0.$

11. $x^2 + 2x - 24 = 0.$

12. $x^2 - 23x + 120 = 0.$

13. $x^3 - 7x^2 + 10x = 0.$

14. $x^2 - 4ax + 3a^2 = 0.$

15. $x^2 - 3ax + 2a^2 = 0.$

16. $x^2 - 3 = \tfrac{1}{6}(x - 3).$

17. $3a^2x^2 + 10ax = 8.$

18. $6x^2 - 5x = 6.$

19. $x^2 + 7x = 0.$

20. $4x - x^2 = -77.$

21. $4x^2 - 11x = -7.$

22. $7x^2 + 15x = -8.$

23. $x^2 - (m - n)x = mn.$

24. $\dfrac{1}{x^2} - \dfrac{1}{x} = 6.$

25. $x^3 - 9x^2 + 8x = 0.$

26. $x^3 - 18x - 3x^2 = 0.$

27. $(x-2)(x^2-7x-8) = 0.$

28. $(x+5)(x^2-6x+5) = 0.$

29. $(x^2 - 2x + 15)(x^2 + 5x + 6) = 0.$

30. Find the roots of $x^3 + x^2 - 10x + 8 = 0$.

By the Remainder Theorem (Art. 156), we find that $x = 1$ satisfies the equation; hence $x - 1$ is a factor of the first member.

Dividing by $x - 1$, and factoring the quotient, we have

$$(x - 1)(x - 2)(x + 4) = 0.$$

$$\therefore x = 1,\ 2,\ \text{or}\ -4.$$

Find the roots of:

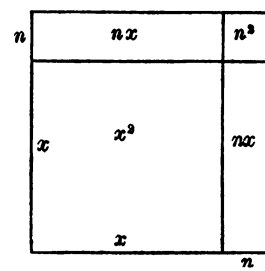
31. $x^3 + 7x^2 + 7x - 15 = 0$. **33.** $x^3 - 2x^2 - 29x + 30 = 0$.

32. $x^3 - 9x^2 + 26x - 24 = 0$. **34.** $(x^2 - 9)(x^3 - 7x + 6) = 0$.

347. Solution by Completing the Square.

It is seen from the identity $(x \pm n)^2 \equiv x^2 \pm 2nx + n^2$ that the last term, n^2, is the square of half the coefficient of x; that is, of half of $2n$. Hence, to make the expression $x^2 \pm 2nx$ a perfect square, we have only to add to it the square of n.

In $x^2 + 2nx + (\ )$, what part of the annexed figure corresponds to the missing term? What needs to be added to complete the square on $x + n$?



We may make any affected quadratic assume the form $x^2 \pm 2nx = m$ by dividing it through by the coefficient of x^2.

Thus, dividing the quadratic $3x^2 \pm 6nx - 9m = 0$ by 3, we have $x^2 \pm 2nx - 3m = 0$, or $x^2 \pm 2nx = 3m$.

1. What must be added to $x^2 + 4x$ to make it a perfect square? To $x^2 + 6x$?

2. How can $x^2 - 8x$ be made a perfect square? $x^2 - 10x$? Is the number added in each case a square? Of what?

3. If 4 is added to both members of the equation $x^2 + 4x = 12$, will they both be perfect squares? Is 4 the square of half the coefficient of x?

4. How can both members of the equation $x^2 - 6x = 7$ be made perfect squares? How is the added term found?

5. How can $5x^2 \pm 10x = 15$ be made to take the form $x^2 \pm 2nx = m$? If $n = 1$ and $m = 3$, how can both members be made perfect squares?

348. From the preceding it appears that the number which must be added to $x^2 + 2nx$ to complete the square *is the square of half the coefficient of* x.

1. Solve the equation $x^2 - 8x = -15$.

Half the coefficient of x is 4. Adding the square of 4, or 16, to each member, we have

$$x^2 - 8x + 16 = -15 + 16 = 1.$$

Extracting the square root, we have

$$x - 4 = \pm 1.$$

$$\therefore x = 4 + 1, \text{ or } 4 - 1,$$

and the roots are 5 and 3.

CHECK. $5^2 - 8 \times 5 = -15$; also $3^2 - 8 \times 3 = -15$.

2. Solve the equation $x^2 - 7x = 8$.

Completing the square, $x^2 - 7x + (\frac{7}{2})^2 = 8 + \frac{49}{4} = \frac{81}{4}$.

Extracting the square root, $x - \frac{7}{2} = \pm \frac{9}{2}$.

$$\therefore x = \tfrac{7}{2} + \tfrac{9}{2}, \text{ or } \tfrac{7}{2} - \tfrac{9}{2},$$

and the roots are 8 and -1.

3. Solve the equation $6x^2 = 6\frac{1}{2} - 10x$.

Transposing $10x$, $6x^2 + 10x = 6\frac{1}{2}$.

Dividing by coefficient of x^2, $x^2 + \frac{10}{6}x = 1\frac{1}{12}$.

Completing square, $x^2 + \frac{10}{6}x + (\frac{5}{6})^2 = 1\frac{1}{12} + \frac{25}{36} = \frac{64}{36}$.

Extracting square root, $x + \frac{5}{6} = \pm \frac{8}{6}$.

$$\therefore x = -\tfrac{5}{6} + \tfrac{8}{6}, \text{ or } -\tfrac{5}{6} - \tfrac{8}{6},$$

and the roots are $\frac{1}{2}$ and $-\frac{13}{6}$.

4. Solve the equation $9x^2 - 22x = 15$.

If the coefficient of x is a *square number*, we may proceed as in the preceding solutions, or we may complete the square without dividing the equation through by this coefficient. This is evident from the identity $(ax \pm n)^2 \equiv a^2x^2 \pm 2anx + n^2$, in which the last term, n^2, is the square of *the quotient obtained by dividing the second term by twice the square root of the first term;* that is, of $2anx \div 2ax$, or n.

Here the square root of $9x^2 = 3x$. Twice $3x = 6x$.

$22x \div 6x = \frac{22}{6}$, or $\frac{11}{3}$. The square of $\frac{11}{3} = 1\frac{3}{9}1$.

Adding $1\frac{3}{9}1$ to both members, we have

$$9x^2 - 22x + 1\tfrac{3}{9}1 = 15 + 1\tfrac{3}{9}1 = 2\tfrac{5}{9}6.$$

Extracting the square root,

$$3x - \tfrac{11}{3} = \pm \tfrac{16}{3}.$$

$$\therefore 3x = \tfrac{11}{3} \pm \tfrac{16}{3} = 9, \text{ or } -\tfrac{5}{3},$$

and

$$x = 3, \text{ or } -\tfrac{5}{9}.$$

If the coefficient of x^2 is not a perfect square, we may multiply the equation by any number that will make the coefficient a square.

5. Solve the equation $2x^2 - 15x = -7$.

Dividing by the coefficient of x^2, $x^2 - \frac{15}{2}x = -\frac{7}{2}$.

Completing the square, $x^2 - \frac{15}{2}x + (\frac{15}{4})^2 = -\frac{7}{2} + \frac{225}{16} = \frac{169}{16}$.

Extracting square root, $x - \frac{15}{4} = \pm \frac{13}{4}$.

$$\therefore x = 7, \text{ or } \tfrac{1}{2}.$$

In cases of this kind fractions may be avoided by multiplying the equation by some number obtained by inspection, or by *multiplying by four times the coefficient of* x^2.

Thus, multiplying by 4×2, or 8,

$$16x^2 - 120x = -56.$$

Completing the square,

$$16x^2 - 120x + 225 = 169.$$

Extracting the square root,

$$4x - 15 = \pm 13,$$

whence

$$4x = 15 \pm 13 = 28, \text{ or } 2.$$

$$\therefore x = 7, \text{ or } \tfrac{1}{2}.$$

Compare the coefficient of x with the square root of the number added to complete the square by this method.

Solve the following, completing the square by the usual method:

6. $x^2 - 2x = 48$.
7. $x^2 + 4x = 32$.
8. $x^2 - 4x = 60$.
9. $x^2 + 6x = 27$.
10. $x^2 - 8x = 48$.
11. $x^2 + 8x = 0$.
12. $x^2 - 8x = 65$.
13. $x^2 - 6x - 7 = 0$.
14. $x^2 - 8x - 9 = 0$.
15. $x^2 + 10x - 24 = 0$.
16. $x^2 - 12x - 28 = 0$.
17. $x^2 + 14x - 15 = 0$.
18. $x^2 - 14x = 32$.
19. $x^2 + 16x = 36$.
20. $x^2 + 20x = 44$.
21. $x^2 - 20x = 69$.
22. $x^2 + 18x = 40$.
23. $x^2 - 22x = 104$.
24. $2x^2 - 6x = 8$.
25. $3x^2 + 12x = 63$.
26. $5x^2 - 10x - 75 = 0$.
27. $3x^2 - 7x - 10 = 0$.
28. $10x^2 - 6x - 4 = 0$.
29. $12x^2 - 11x + 2 = 0$.

Solve the following, completing the square by method used in Ex. 4:

30. $4x^2 - 11x = 45$.
31. $9x^2 + 8x = 52$.
32. $16x^2 - 30x = 54$.
33. $25x^2 - 45x = 10$.
34. $25x^2 - 48x - 81 = 0$.
35. $36x^2 - 25x - 11 = 0$.
36. $36x^2 + 10x - 124 = 0$.
37. $100x^2 + 225x - 1375 = 0$.

Solve the following, avoiding fractions by method used in Ex. 5:

38. $3x^2 - 2x = 8$.
39. $2x^2 + 5x = 7$.
40. $2x^2 + 7x = -3$.
41. $3x^2 - 14x = 5$.
42. $6x^2 + 5x = 56$.
43. $3x^2 + 2x = 5$.
44. $2x^2 + 7x = 4$.
45. $3x^2 - 13x = 10$.
46. $3x^2 + 15x = 42$.
47. $5x^2 + 11x = 12$.
48. $6x^2 - 20x = 16$.
49. $5x^2 - 3x - 2 = 0$.
50. $7x^2 - 11x - 6 = 0$.
51. $2x^2 - 9x - 56 = 0$.
52. $3x^2 + 10x - 32 = 0$.
53. $6x^2 - 17x + 12 = 0$.

349. Solution by Formula.

Every affected quadratic can be reduced to the general form

$$ax^2 + bx + c = 0,$$

in which a may represent any *positive* number except 0, b any positive or negative number not 0, and c any number whatever.

This general equation may be solved as follows:

Transposing c, $\quad ax^2 + bx = -c.$

Multiplying by $4a$, $\quad 4a^2x^2 + 4abx = -4ac.$

Completing square, $\quad 4a^2x^2 + 4abx + b^2 = b^2 - 4ac.$

Extracting root, $\quad 2ax + b = \pm\sqrt{b^2 - 4ac},$

or $\quad 2ax = -b \pm \sqrt{b^2 - 4ac}.$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (1)$$

The roots of any affected quadratic equation can be found by substituting in this general formula the coefficient of x^2 for a, that of x for b, and the absolute term for c.

If $a = 1$, the above formula becomes

$$x = \frac{-b \pm \sqrt{b^2 - 4c}}{2}, \qquad (2)$$

which is a more convenient form in which to make the substitution, when the quadratic is of the type $x^2 + bx + c = 0$; that is, when the coefficient of x^2 is unity.

1. Solve the equation $2x^2 + 5x + 2 = 0.$

$$x = \frac{-5 \pm \sqrt{25 - 8 \times 2}}{4} = \frac{-5 \pm 3}{4} = -\tfrac{1}{2}, \text{ or } -2.$$

Solve by substituting in the formula:

2. $2x^2 + 3x - 2 = 0.$
3. $3x^2 - 5x - 12 = 0.$
4. $3x^2 + 11x = -6.$
5. $2x^2 + 3x - 9 = 0.$
6. $4x^2 - 3x = 2.$
7. $6x^2 - x - 15 = 0.$
8. $3x(x - 2) + 2 = 0.$
9. $(x + 2)^2 + 8 = 0.$
10. $7x(x + 1) = 14.$
11. $5x(x - 3) = 2(9 - x^2).$

350. In solving a quadratic equation by the first method, the equation is separated into factors. In solving by the method of completing the square, the members are separated into two equal factors by extracting the square root. The formulas are obtained by solving the general equation by one of the methods just mentioned. In solving by either of them, the solution is made to depend upon forming from the quadratic two linear equations to which it is equivalent.

351. (1) If the quadratic can be readily *factored by inspection, it is best to solve by the factoring method.*

(2) If the equation is *not easily factored, solve by completing the square.*

If the coefficient of x^2 is a square number, it is usually best to complete the square by the method used in Ex. 4, Art. 348. If fractions would arise in completing the square, it is preferable to multiply by 4 times the coefficient of x^2, as in Ex. 5. This method is usually to be preferred in solving literal equations. The method given in Art. 348 is the best general method, and is the one usually employed.

(3) After the student has become thoroughly familiar with the preceding methods, much time is saved by *using the formula at once.*

1. Solve the equation $\dfrac{x+1}{x-1}+\dfrac{x+3}{x-3}=\dfrac{11}{2}$.

The L. C. M. of the denominators is $2(x-1)(x-3)$. Multiplying by this, we have

$$2(x+1)(x-3)+2(x-1)(x+3)=11(x-1)(x-3).$$

$$\therefore\ -7x^2+44x=45,$$

or $$7x^2-44x=-45.$$

Dividing by the coefficient of x^2, $x^2-\frac{44}{7}x=-\frac{45}{7}$.

Completing the square, $x^2-\frac{44}{7}x+(\frac{22}{7})^2=-\frac{45}{7}+\frac{484}{49}=\frac{169}{49}$.

Extracting the square root, $x-\frac{22}{7}=\pm\frac{13}{7}$.

$$\therefore\ x=\tfrac{22}{7}\pm\tfrac{13}{7}=5,\ \text{or}\ \tfrac{9}{7}.$$

The principles already given for solving fractional equations which lead to simple equations, apply also in the solution of fractional equations leading to quadratic equations.

2. Solve the equation $ax^2 - a^2x + abx - a^2b = 0$.

This equation may be written

$$ax^2 - (a^2 - ab)x = a^2b.$$

Multiplying by 4 times the coefficient of x^2,

$$4a^2x^2 - 4a(a^2 - ab)x = 4a^3b.$$

Completing the square by adding $(a^2 - ab)^2$,

$$4a^2x^2 - (\quad) + (a^2 - ab)^2 = 4a^3b + (a^2 - ab)^2 = (a^2 + ab)^2.$$

Extracting the square root,

$$2ax - (a^2 - ab) = \pm(a^2 + ab),$$

whence $$2ax = a^2 - ab \pm (a^2 + ab) = 2a^2, \text{ or } -2ab.$$

$$\therefore x = a, \text{ or } -b.$$

It is seen that the principles that apply in the solution of numerical quadratic equations are also applicable in the case of literal quadratic equations.

3. Solve the equation $x^2 + ax - bx = ab$.

This equation, written in the form

$$x^2 + (a - b)x - ab = 0,$$

shows at once that its first member is the product of two binomials. The first term of each is evidently x. Since $-ab$ is the product of the second terms, and $a - b$ is their algebraic sum, $+a$ and $-b$ are those terms. Hence, the equation may be written

$$(x + a)(x - b) = 0.$$

$$\therefore x = -a, \text{ or } b.$$

Solve the following equations by the most convenient method, checking results:

4. $x^2 - 6x = 5x$.

5. $x^2 - 4x = -8$.

6. $x^2 + 4ax = 5a^2$.

7. $x^2 - 2ax = 3a^2$.

8. $3x^2 - 2x = 8$.

9. $5x^2 + 6x = 27$.

10. $5x^2 - 7x = 24$.

11. $8x^2 - 30x - 50 = 0$.

12. $x^2 - 6nx = 7n^2$.

13. $x^2 + 8ax = 0$.

14. $7x^2 - 20x - 32 = 0$.

15. $10x^2 + 42x - 40 = 0$.

16. $3x^2 - 12x = 15$.

17. $2x^2 + 10x = 12$.

18. $x^2 + n^2 = nx$.

19. $5x^2 + 2x = 3$.

20. $7x^2 - 12x = 64$.
21. $3x^2 - 22x = 45$.
22. $2x^2 - 12x + 16 = 0$.
23. $7x^2 + 5x - 38 = 0$.
24. $6x^2 + 5x - 14 = 0$.
25. $9x^2 + 2x - 75 = 0$.
26. $2x^2 - 14x - 36 = 0$.
27. $6x^2 + 9x - 42 = 0$.
28. $7x^2 - 28x - 84 = 0$.
29. $7x^2 - 16x = -4$.
30. $x + 2 - 6x^2 = 0$.
31. $2x(4x + 1) - 3 = 0$.
32. $1 + 2x(3 + 4x) = 0$.
33. $32(x^2 - x) = 3 - 16x^2$.
34. $11x + 12(1 - 3x^2) = 0$.
35. $12x(x - 1) - 45 = 0$.
36. $3x(2x - 7) - 12 = 0$.
37. $(x - 1)(3x - 2) - 14 = 0$.
38. $3x^2 - 4 = (x + 2)(2x + 1)$.
39. $x^2 - 4abx - (a^2 - b^2)^2 = 0$.
40. $2x^2 + (2n - a)x - an = 0$.
41. What are the roots of $ax^2 + bx + c = 0$, if $b^2 = 4ac$?
42. $x^2 - 4x + b = 0$.
43. $x^2 - px + q = 0$.
44. $x^2 - (a + b)x = -ab$.
45. $x^2 + mx = n^2 + m^2$.
46. $x^2 - ax - a - 1 = 0$.
47. $x^2 - 2cx = 2c + 1$.
48. $4a^2x^2 + 2ax - 2 = 0$.
49. $x^2 + n^2 = 4(m^2 + nx)$.
50. $2x^2 - 8ax = 2 - 2a^2$.
51. $x^2 + 2(b + 8)x = -32b$.
52. $\dfrac{2x + 1}{x + 7} + \dfrac{4x + 1}{x - 7} - \dfrac{6}{x^2 - 49} = 0$.
53. $\dfrac{3}{x - 1} + \dfrac{1}{x + 1} = 3\frac{1}{3}$.
54. $\dfrac{x - 6}{x + 30} = \dfrac{3}{x - 6}$.
55. $\dfrac{x + 5}{2x + 1} = 1 - \dfrac{x - 6}{x - 2}$.
56. $\dfrac{x + 2}{x + 1} + \dfrac{x + 1}{x + 2} = \dfrac{13}{6}$.
57. $\dfrac{60}{x + 4} - \dfrac{x}{x - 6} = 1$.
58. $\dfrac{x + 1}{x + 2} = \dfrac{x + 3}{x + 4} + \dfrac{8}{3}$.
59. $\dfrac{x + 1}{x - 2} - \dfrac{3x - 1}{x + 2} = 0$.
60. $\dfrac{x + 3}{x - 3} - \dfrac{x + 4}{x - 2} = 3$.
61. $\dfrac{5x + 8}{x + 4} - \dfrac{x + 3}{x - 2} = 0$.
62. $\dfrac{a^2}{1 + a} = \dfrac{x^2}{1 + x}$.
63. $\dfrac{a}{x} + \dfrac{x}{a} = \dfrac{3}{2}$.
64. $\dfrac{3}{4a^2} + \dfrac{x}{a} = x^2$.

65. $\frac{2a}{x}+2=\frac{a}{x-a}$.

66. $\frac{1}{x}-\frac{a}{b}=\frac{b}{a}-x$.

67. $\frac{3}{x^2-1}-\frac{1}{2(x+1)}=\frac{1}{4}$.

68. $\frac{2}{x-4}+\frac{3}{x-2}=\frac{2}{x-1}$.

69. $\frac{1-x}{2+x}=\frac{2}{5(x-2)}-\frac{1+x}{x^2-4}$.

70. $4.4+\frac{3}{x+2}=\frac{7}{x^2-4}$.

71. $\frac{5x-1}{x+3}+\frac{1}{8}+\frac{7x^2-106}{8(x^2-9)}=0$.

72. $\frac{3+x^2}{a^2-x^2}+\frac{1}{a+x}=\frac{1}{a-x}$.

73. $\frac{x+1}{x-3}+\frac{x+9}{x^2-9}=\frac{24}{x+3}$.

74. $\frac{2(x-1)}{5(x+1)}-\frac{(x+1)(x-1)}{x^2+1}=0$.

75. $\frac{8x-1}{1-x^2}+\frac{1+2x}{1-x}=\frac{12}{1+x}$.

76. $\frac{x-8}{x-3}-\frac{1}{2}=\frac{80}{x^2-9}-\frac{x-5}{x+3}$.

77. $\frac{x}{x+1}+m=\frac{x}{x-1}$.

78. $\frac{a^2x^2}{b^2}-\frac{b^2}{c^2}=\frac{2ax}{c}$.

79. $x^2+\frac{m^2-n^2}{mn}x-1=0$.

80. $ax^2+\frac{b^2}{a}=2bx+\frac{c^2}{a}$.

81. $\frac{3(2x+1)}{x-2}-\frac{13-3x}{x-1}=19$.

82. $\frac{n-x}{x-m}-\frac{b}{a}+\frac{x-m}{n-x}=0$.

83. $1+\frac{n-1}{x^2-2ax^2+a^2x^2}=\frac{n}{ax-x}$.

84. $\frac{a-x}{a+x}-\frac{a^2}{a^2-x^2}=\frac{2(a+x)}{a-x}$.

85. $1+\frac{m-2n}{x+2n}=\frac{x+13m+3n}{5m-3n-x}$.

86. $x^2-\frac{mx}{m+n}+\frac{nx}{m+n}-\frac{mn}{(m+n)^2}=0$.

EQUATIONS IN THE QUADRATIC FORM

352. When the unknown number occurs in but two terms of an equation, and its exponent in one is twice that in the other, the equation is in the *quadratic form.*

Thus, $3x^4+6x^2=9$, $x+5\sqrt{x}=14$, $(x^2+3)^2+6(x^2+3)=40$, and $ax^{2n}+bx^n+c=0$ are in the quadratic form.

353. The methods used to solve equations in the quadratic form are the same as those employed in the solution of quadratics.

1. Solve the equation $x^6 + 7x^3 = 8$.

This is a quadratic in x^3, being the same as

$$(x^3)^2 + 7(x^3) - 8 = 0.$$

Factoring, $$(x^3 + 8)(x^3 - 1) = 0.$$

$$\therefore x^3 = -8, \text{ or } 1.$$

$$\therefore x = -2, \text{ or } 1.$$

Equations of higher degree than the second are often only *partially* solved by the quadratic method. This equation is of the 6th degree, and hence has six roots. The other four roots may be obtained by factoring $x^3 + 8$ and $x^3 - 1$. By the Remainder Theorem, or by trial, we find that $x^3 + 8$ is divisible by $x + 2$, and $x^3 - 1$ is divisible by $x - 1$.

$$\therefore (x - 1)(x^2 + x + 1) = 0, \text{ and } (x + 2)(x^2 - 2x + 4) = 0.$$

From $x^2 + x + 1 = 0$, we get $x = -\frac{1}{2} \pm \frac{1}{2}\sqrt{-3}$;

and from $x^2 - 2x + 4 = 0$, we get $x = 1 \pm \sqrt{-3}$.

2. Solve the equation $x^2 + \frac{1}{x^2} + x + \frac{1}{x} = 4$.

Adding 2 to each member, $$\left(x + \frac{1}{x}\right)^2 + \left(x + \frac{1}{x}\right) = 6.$$

Completing the square, etc., $$x + \frac{1}{x} + \frac{1}{2} = \pm\frac{5}{2}.$$

Transposing and clearing, $$x^2 + 1 = 2x, \text{ or } -3x.$$

$$\therefore x = 1, \text{ or } \tfrac{1}{2}(-3 \pm \sqrt{5}).$$

3. Solve $\sqrt{x^3} - 5\sqrt[4]{x^3} = 24$.

Using fractional exponents, we have $$x^{\frac{3}{2}} - 5x^{\frac{3}{4}} = 24.$$

Completing the square, $$x^{\frac{3}{2}} - 5x^{\frac{3}{4}} + (\tfrac{5}{2})^2 = 24 + \tfrac{25}{4} = 12\tfrac{1}{4}.$$

Extracting the square root, $$x^{\frac{3}{4}} - \tfrac{5}{2} = \pm 3\tfrac{1}{2},$$

whence $$x^{\frac{3}{4}} = \tfrac{5}{2} \pm 3\tfrac{1}{2} = 8, \text{ or } -3.$$

Extracting the cube root, $$x^{\frac{1}{4}} = 2, \text{ or } -3^{\frac{1}{2}}.$$

Raising to the fourth power, $$x = 16, \text{ or } 3\sqrt[3]{3}.$$

4. Solve the equation $(x^2+x-6)(x^2+x-20)=120$.

If $y = x^2+x-6$,

then $y(y-14)=120$.

$\therefore y=20$, or -6.

$\therefore x^2+x-6=20$, and $x^2+x-6=-6$.

$\therefore x=\frac{1}{2}(-1\pm\sqrt{105})$, 0, or -1.

5. Solve $2x^2+3x-5\sqrt{2x^2+3x+9}=-3$.

If we add 9 to each side, this equation will be in the quadratic form, thus:

$$2x^2+3x+9-5(2x^2+3x+9)^{\frac{1}{2}}=6.$$

Completing the square, we have

$$2x^2+3x+9-5(2x^2+3x+9)^{\frac{1}{2}}+\tfrac{25}{4}=6+\tfrac{25}{4}=\tfrac{49}{4}.$$

Extracting the square root,

$$(2x^2+3x+9)^{\frac{1}{2}}-\tfrac{5}{2}=\pm\tfrac{7}{2},$$

or $$(2x^2+3x+9)^{\frac{1}{2}}=\tfrac{5}{2}\pm\tfrac{7}{2}=6, \text{ or } -1.$$

Squaring both members, $2x^2+3x+9=36$, or 1.

From these two equations we readily find the roots to be 3, $-4\frac{1}{2}$, or $\frac{-3\pm\sqrt{-55}}{4}$.

For convenience we may let m stand for $(2x^2+3x+9)^{\frac{1}{2}}$, and m^2 for $2x^2+3x+9$. We should then have

$$m^2-5m=6, \text{ whence } m=6, \text{ or } -1.$$

6. Solve $x^4-8x^3+22x^2-24x=0$.

An inspection of the left member tells us that it is almost a square. By extracting the square root we find that it lacks 9 of being a square. Hence, adding 9 to each side, we have

$$x^4-8x^3+22x^2-24x+9=9,$$

or $$(x^2-4x+3)^2=9.$$

$$\therefore x^2-4x+3=\pm 3.$$

$$\therefore x=4,\ 0,\ 2\pm\sqrt{-2}.$$

Solve the following:

7. $x^4-5x^2=-4$.

8. $x^4-16x^2=225$.

9. $x^4-21x^2-100=0$.

10. $x^6-9x^3+8=0$.

11. $x^6 - 35x^3 + 216 = 0.$
12. $2x^4 - 3x^2 - 12 = 0.$
13. $8x^4 - 3x^2 + \frac{1}{4} = 0.$
14. $x^6 + 6x^3 - 16 = 0.$
15. $x^3 - x^{\frac{3}{2}} - 56 = 0.$
16. $x^{\frac{1}{2}} + x^{\frac{1}{4}} - 6 = 0.$
17. $x^{\frac{1}{2}} + x^{-\frac{1}{2}} - 3\frac{1}{3} = 0.$
18. $x + 5\sqrt{x} - 24 = 0.$
19. $x^{\frac{2}{3}} - 4x^{\frac{1}{3}} + 4 = 0.$
20. $1 + 6x^{\frac{1}{2}} = x^{-\frac{1}{2}}.$
21. $8x^{-\frac{6}{5}} = 2 + 15x^{-\frac{3}{5}}.$
22. $10x^{2n} + 6x^n - 13\frac{1}{2} = 0.$
23. $2x^2 + \sqrt{2x^2 + 1} - 11 = 0.$
24. $(x-3) - 5\sqrt{x-3} + 4 = 0.$
25. $(x^2+1)^2 + 3x^2 + 3 = 0.$
26. $2x^{-\frac{4}{3}} - 21x^{-\frac{2}{3}} = -27.$
27. $(x-2)^2 + 7(x-2) - 8 = 0.$
28. $(x^2+1)^2 + 6(x^2+1) - 55 = 0.$
29. $(x^2-x)^2 - 3(x^2-x) - 108 = 0.$
30. $x^2 - 4x + 5\sqrt{x^2 - 4x + 4} = 20.$
31. $\sqrt{x+12} + \sqrt[4]{x+12} = 6.$
32. $2x^2 + 4x + 3\sqrt{2x^2 + 4x + 9} = 31.$
33. $4x^2 + 12x + 6\sqrt{4x^2 + 12x - 2} + 3 = 0.$
34. $(x+10)^{\frac{1}{2}} + 3(x+10)^{\frac{1}{4}} - 10 = 0.$
35. $(x-1)^4 - 8(x-1)^2 - 9 = 0.$
36. $(x^2-5)^2 - 5(x^2-5) + 4 = 0.$
37. $(x^2+2x)^2 + 9(x^2+2x) - 36 = 0.$
38. $3(x^2+5x) - 2(x^2+5x+1)^{\frac{1}{2}} - 2 = 0.$
39. $4x^4 - 20x^3 + 23x^2 + 5x - 6 = 0.$
40. $\dfrac{x}{a+x} + \dfrac{a}{\sqrt{a+x}} = \dfrac{2a^2}{x}.$
41. $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0.$
42. $x^4 + 14x^3 + 47x^2 - 14x - 48 = 0.$
43. $3(x-2)^{\frac{4}{3}} - 7(x-2)^{\frac{2}{3}} + 4 = 0.$
44. $(x^2-7)^{\frac{1}{2}} + (x^2-7)^{-\frac{1}{2}} - 3\frac{1}{3} = 0.$

354. Radical Equations.

The principles already given for solving quadratic equations which lead to simple equations apply also in the solution of radical equations which lead to quadratics.

1. Solve the equation $\sqrt{x+1}+\sqrt{x-2}=\sqrt{3x}$.

$$\sqrt{x+1}+\sqrt{x-2}=\sqrt{3x}.$$

Squaring, $x+1+2\sqrt{(x+1)(x-2)}+x-2=3x,$

or $2\sqrt{(x+1)(x-2)}=x+1.$

Squaring, $4(x+1)(x-2)=(x+1)^2,$

or $4x^2-4x-8=x^2+2x+1.$

$\therefore x^2-2x-3=0.$

Factoring, $(x-3)(x+1)=0.$

$\therefore x=3,$ or $-1.$

CHECK. $\sqrt{4}+\sqrt{1}=\sqrt{9};\ \sqrt{0}+\sqrt{-3}=\sqrt{-3}.$

2. Solve the equation $\sqrt{x+2}+\sqrt{2x+2}=x.$

Transposing, $\sqrt{2x+2}=x-\sqrt{x+2}.$

Squaring, $2x+2=x^2-2x\sqrt{x+2}+x+2,$

or $2x\sqrt{x+2}=x^2-x,$

or, dividing by x, $2\sqrt{x+2}=x-1.$

Squaring, $4x+8=x^2-2x+1.$

$\therefore x^2-6x-7=0.$

Factoring, $(x-7)(x+1)=0.$

$\therefore x=7,$ or $-1.$

CHECK. For $x=7$, $\sqrt{9}+\sqrt{16}=7$; for $x=-1$, $\sqrt{1}+\sqrt{0}=-1$, or $(\pm 1)=-1$, which is true in $(-1)=-1$, but it is not true if we consider only the positive root of the radicals. The root -1 is commonly called extraneous (Art. 324).

3. Solve the equation $x^2+x-2=2\sqrt{x^2+x-2}.$

Dividing by $\sqrt{x^2+x-2}$, we have

$$\sqrt{x^2+x-2}=2.$$

Putting in general form, $x^2+x-6=0.$

Solving by formula,

$$x=\frac{-1\pm\sqrt{1+24}}{2}=\frac{-1\pm 5}{2}=-3,\text{ or }+2.$$

Since one or more roots may be lost by dividing the equation by a factor containing the unknown number (Art. 204, 3), the factor used as a divisor should be placed equal to 0, and solved, and these values substituted in the original equation to test for lost roots.

Placing $\sqrt{x^2+x-2}$ equal to 0 and squaring both members, we have $x^2+x-2=0$, the solution of which gives the roots 1 and -2. These roots satisfy the original equation, but were lost by dividing by $\sqrt{x^2+x-2}$.

We may also find the four roots of the equation by freeing it of radicals and factoring, or by proceeding as follows:

The equation may be written,

$$x^2+x-2-2\sqrt{x^2+x-2}=0.$$

If we let $y=\sqrt{x^2+x-2}$,

then $y^2-2y=0$.

$\therefore y(y-2)=0$.

$\therefore y=0$, or 2.

Hence $x^2+x-2=0$, or 4.

If $x^2+x-2=0$,

then $(x+2)(x-1)=0$.

$\therefore x=1$, or -2.

If $x^2+x-2=4$,

then $x^2+x=6$.

Putting in general form, $x^2+x-6=0$.

By formula,

$$x=\frac{-1\pm\sqrt{1+24}}{2}=\frac{-1\pm5}{2}=-3, \text{ or } +2.$$

Solve the following equations, rejecting roots commonly called extraneous:

4. $\sqrt{4x+5}+5=2x$.

5. $3x-1=2\sqrt{x}$.

6. $x+7-\sqrt{x+7}=12$.

7. $\sqrt{3x+10}-\sqrt{2x-1}=2$.

8. $(\sqrt{x}-2)^2=\frac{\sqrt{x}+2}{5}$.

9. $2\sqrt{5x^2-2x+9}=5x$.

10. $\sqrt{x+8}=\sqrt{3x+1}-1$.

11. $2\sqrt{1+8x}-3=\sqrt{x+8}$.

12. $\frac{\sqrt{x^2+11}}{\sqrt{x^2-9}}=\frac{\sqrt{x^2-9}}{2\frac{2}{3}}$.

13. $x+2+\sqrt{2x-1}=2x$.

14. $\sqrt{2x-1}+\sqrt{x-1}=\sqrt{5x}$.

15. $\sqrt{3+x}+\sqrt{x}=\sqrt{8+x}$.

16. $\dfrac{2\sqrt{x}-1}{\sqrt{x}-2}=\dfrac{3\sqrt{x}+11}{\sqrt{x}+1}$.

17. $(7-\sqrt{x})^2=5(3+\sqrt{x})$.

18. $\sqrt{5+x}-2\sqrt{x}=1$.

19. $\sqrt{10-4x}-\sqrt{x}=\sqrt{6+x}$.

20. $\sqrt{5x-\sqrt{3x+4}}=4$.

21. $\sqrt{7+x}+2\sqrt{x}=\dfrac{15}{\sqrt{7+x}}$.

22. $2\sqrt{3x-2}-2\sqrt{x+3}=\sqrt{x-2}$.

23. $\sqrt{3x+7}-\sqrt{2x+3}=\sqrt{x-2}$.

24. $\sqrt{2x+1}=\sqrt{x+3}+\sqrt{x-2}$.

25. $4\sqrt{x-1}-\sqrt{x+4}=\sqrt{x+20}$.

26. $\sqrt{2x+9}-\sqrt{x-4}=\sqrt{x+1}$.

27. $\dfrac{5x+\sqrt{30x-x^2}}{5x-\sqrt{30x-x^2}}=4$.

28. $\dfrac{\sqrt{2x}-3}{9-2\sqrt{2x}}=\dfrac{3+\sqrt{2x}}{3\sqrt{2x}-5}$.

29. $\sqrt{\dfrac{3x-3}{3x+4}}+\sqrt{\dfrac{3x+4}{3x-3}}=\dfrac{25}{12}$.

30. $\sqrt{x^2+1}+\dfrac{3}{\sqrt{x^2+1}}=4$.

ROOTS OF THE QUADRATIC EQUATION

355. Number of Roots.

It has been seen in Art. 206 that a *linear* equation has but one root, and since a quadratic expression, including the general equation $ax^2+bx+c=0$, has two, and only two, *linear factors*, it follows that *every quadratic equation has two roots, and only two.*

For instance, if the general quadratic equation $ax^2+bx+c=0$ is written in the form $x^2+\dfrac{b}{a}x+\dfrac{c}{a}=0$, then by *adding* and *subtracting* $\left(\dfrac{b}{2a}\right)^2$, or $\dfrac{b^2}{4a^2}$, we get

$$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}-\frac{b^2}{4a^2}+\frac{c}{a}=0,$$

or

$$\left(x+\frac{b}{2a}\right)^2-\left(\frac{b^2}{4a^2}-\frac{c}{a}\right)=0.$$

Since any number is the square of its square root, we see that the left member is in the form of the difference of two squares. Factoring, we get

$$\left(x+\frac{b}{2a}+\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}\right)\left(x+\frac{b}{2a}-\sqrt{\frac{b^2}{4a^2}-\frac{c}{a}}\right)=0.$$

Each *linear* factor, equated to 0, gives one of the two roots

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

This is also seen to be the case from the solution of the above equation as given in Art. 349, for the roots being

$$\frac{-b+\sqrt{b^2-4ac}}{2a} \text{ and } \frac{-b-\sqrt{b^2-4ac}}{2a},$$

it follows that the value of x *must be* one or the other of these two numbers.

It can also be shown that an equation of the third degree has three roots, one of the fourth degree four roots, and so on. In general, an equation of the nth degree has n roots.

356. Nature of the Roots.

It is not necessary to solve a quadratic equation in order to know the nature of its roots; that is, whether both are *real* and *unequal*, both *real* and *equal*, or both *imaginary* and *unequal*.

We have seen that a solution of the general equation $ax^2+bx+c=0$, in which a, b, and c are limited to *real* values, gives the roots

$$r_1=-\frac{b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}, \text{ and } r_2=-\frac{b}{2a}-\frac{\sqrt{b^2-4ac}}{2a}.$$

It is evident from an inspection of these roots that their *nature* depends wholly upon the expression under the radical sign; that is, upon whether b^2-4ac is positive, negative, or zero.

1. If $b^2-4ac>0$, the roots are *real* and *unequal*.

For, since b^2-4ac is *positive*, its square root can be found either exactly or approximately. Hence the roots are *real*.

Since $b^2 - 4ac$ is not 0, the term $-\frac{b}{2a}$ will be increased for one root and diminished for the other. Hence the roots are *unequal.*

If $\sqrt{b^2 - 4ac}$ can be found exactly, the roots are *rational;* otherwise they are *surds.*

2. If $b^2 - 4ac = 0$, the roots are *real* and *equal.*

For, since $\sqrt{b^2 - 4ac} = 0$, both roots become $-\frac{b}{2a}$. Hence they are *equal* and *real.* In this case $-\frac{b}{2a}$ is called a *double root.*

3. If $b^2 - 4ac < 0$, the roots are *imaginary.*

For, since $b^2 - 4ac$ is *negative,* its square root can only be *indicated.* Hence each root has an *imaginary* part as well as a real part. Since the real parts are the same, and the imaginary parts the same but with opposite signs, the roots are *conjugate complex numbers. Complex roots enter in pairs.*

357. We can now determine the nature of the roots of any given quadratic equation without solving it. We need only find the value of $b^2 - 4ac$.

1. Find the nature of the roots of $x^2 - 3x - 4 = 0$.

Here $a = 1, b = -3, c = -4.$

$\therefore b^2 - 4ac = 9 - (-16) = 25$, a *positive* number.

Therefore the roots are real and unequal; and since $b^2 - 4ac$ is a perfect square, the roots are *rational.*

2. Find the nature of the roots of $x^2 - 8x + 16 = 0$.

Here $a = 1, b = -8, c = 16.$

$\therefore b^2 - 4ac = 64 - 64 = 0.$

Therefore the roots are real and equal.

3. Find the nature of the roots of $x^2 - 4x + 5 = 0$.

Here $a = 1, b = -4, c = 5.$

$\therefore b^2 - 4ac = 16 - 20 = -4$, a *negative* number.

Therefore the roots are complex. A solution of the equation gives as roots the conjugate complex numbers $2 + \sqrt{-1}$ and $2 - \sqrt{-1}$.

4. What is the nature of the roots of $3x^2 + 0x - 9 = 0$?

Here $a = 3,\ b = 0,\ c = -9.$

$$\therefore\ -\frac{b}{2a} = 0, \text{ and } b^2 - 4ac = 108.$$

Therefore the roots are equal but opposite, being $\pm\frac{\sqrt{108}}{6} = \pm\sqrt{3}.$

5. For what values of m does the equation

$$(2m - 3)x^2 + mx + (m - 1) = 0$$

have its two roots equal?

Here $a = 2m - 3,\ b = m,\ c = m - 1.$

Since the roots are to be equal, we must have $b^2 - 4ac = 0$, or $m^2 - 4(2m - 3)(m - 1) = 0.$

Solving this equation, we find

$$m = 2, \text{ or } \tfrac{6}{7}.$$

Using these values of m we have the equations $x^2 + 2x + 1 = 0$, and $9x^2 - 6x + 1 = 0$, each of which has its two roots equal.

Without solving, find the nature of the roots of the following equations:

6. $x^2 + 4x - 12 = 0.$

7. $x^2 - 2x + 3 = 0.$

8. $x^2 - 5x + 2 = 0.$

9. $x^2 - 6x - 27 = 0.$

10. $x^2 + 2x + 3 = 0.$

11. $x^2 + 3x - 5 = 0.$

12. $x^2 + x - 2 = 0.$

13. $3x^2 + 7x + 2 = 0.$

14. $4x^2 - 4x + 1 = 0.$

15. $2x^2 - 9x + 4 = 0.$

16. $8x^2 - 2x - 25 = 0.$

17. $4x^2 + 16x + 7 = 0.$

18. $6x^2 - 7x - 5 = 0.$

19. $9x^2 - 18x + 8 = 0.$

For what values of m are the roots of the following equations equal?

20. $2x^2 + mx + 1 = 0.$

21. $mx^2 + 6x + 1 = 0.$

22. $3x^2 + 9x + m = 0.$

23. $mx^2 + mx + 1 = 0.$

24. $(m + 1)x^2 + (m - 1)x + (m + 1) = 0.$

25. $2mx^2 + (5m + 2)x + (4m + 1) = 0.$

358. Relation of Roots to Coefficients.

We have already seen that the roots of the general equation $ax^2 + bx + c = 0$, or $x^2 + \frac{b}{a}x + \frac{c}{a} = 0$, are

$$r_1 = -\frac{b}{2a} + \frac{\sqrt{b^2 - 4ac}}{2a},$$

$$r_2 = -\frac{b}{2a} - \frac{\sqrt{b^2 - 4ac}}{2a}.$$

The sum of these roots is $-\frac{b}{a}$. (1)

The product of the roots is

$$\frac{(-b)^2 - (b^2 - 4ac)}{4a^2} = \frac{c}{a}. \quad (2)$$

Comparing these results with the given equation, we see that, in a quadratic *where the coefficient of x^2 is unity*,—

(*a*) The *sum* of the roots is equal to the coefficient of x *with its sign reversed;*

(*b*) The *product* of the roots is equal to the *constant term.*

Thus, the roots of the equation $x^2 - 7x + 10 = 0$ are 5 and 2. Their sum is + 7, their product + 10.

It is important to observe that these relations furnish a valuable check upon solutions, since any solution which does not obey these laws is incorrect.

Solve the following, checking results by the above method:

1. $x^2 - x - 1 = 0$.
2. $x^2 - 2x - 8 = 0$.
3. $x^2 - x - 6 = 0$.
4. $2x^2 - x - 15 = 0$.
5. $2x^2 - 3x - 5 = 0$.
6. $3x^2 - 12x - 15 = 0$.

359. Formation of Equations from Given Roots.

When the roots of a quadratic equation are given, the relations expressed in the preceding article enable us to form the equation.

Form the equation whose roots are 1 and -2.

Here the *sum* of the roots is $1 - 2 = -1$.

$\therefore$ The coefficient of x is $+1$.

The product of the roots is $1 \times -2 = -2$.

$\therefore$ -2 is the constant term.

Hence, the equation is $x^2 + x - 2 = 0$, assuming, as we always may, that the coefficient of x^2 is 1.

360. An equation may also be formed from its roots in the manner indicated in the following:

If the roots of an equation are a and $-b$, we have the two equations

$$x = a \text{ and } x = -b,$$

or

$$x - a = 0 \text{ and } x + b = 0.$$

We know by Art. 202 that these two equations are jointly equivalent to the equation whose given roots are a and $-b$. Hence, $(x - a)(x + b) = 0$ is that equation in factored form, and the equation is $x^2 - (a - b)x - ab = 0$.

1. Form the quadratic equation whose roots are 2 and -3.

The factored form of the equation is

$$(x - 2)(x + 3) = 0,$$

whence $x^2 + x - 6 = 0$ is the equation.

2. Form the quadratic equation whose roots are $-1 \pm \sqrt{3}$.

$$(x + 1 - \sqrt{3})(x + 1 + \sqrt{3}) = 0,$$

whence

$$x^2 + 2x - 2 = 0.$$

Form the equations whose roots are:

3. 3, -2.
4. -1, 2.
5. 2, 3.
6. 3, 3.
7. 0, 2.
8. $-\frac{1}{2}$, $\frac{1}{3}$.
9. $-\frac{1}{3}$, 4.
10. b, $-a$.
11. $\frac{m}{n}$, $\frac{n}{m}$.
12. $1 + \sqrt{3}$, $1 - \sqrt{3}$.
13. $2 - \sqrt{2}$, $2 + \sqrt{2}$.
14. $\frac{1}{2}\sqrt{-5}$, $-\frac{1}{2}\sqrt{-5}$.
15. $\frac{1}{3} - \frac{1}{2}\sqrt{7}$, $\frac{1}{3} + \frac{1}{2}\sqrt{7}$.
16. $3 \pm 2\sqrt{-1}$.
17. $-a \pm 2b\sqrt{-1}$.

PROBLEMS

361. Owing to the number and nature of the roots of a quadratic equation, one or both of them may be excluded by the *conditions* of the problem whose solution gives rise to the equation. Imaginary roots indicate that the problem is impossible. A negative root may suggest a similar problem with some of the conditions changed. A fractional root may show that a solution of the problem is physically impossible. It is therefore necessary to determine whether either or both of the roots satisfy the conditions of the problem.

1. The sum of two numbers is 12 and their product is 27. What are the numbers?

Let x and $12 - x$ represent the numbers; then $x(12 - x) = 27$, whence $x = 3$, or 9.

Here both values are applicable to the problem, for if $x = 3$, $12 - x = 9$; and if $x = 9$, $12 - x = 3$. That is, the two numbers are the roots of the equation.

2. The sum of the squares of two consecutive numbers is 181. What are the numbers?

Let x = one number, $x + 1$ = the other.

Then $$x^2 + (x + 1)^2 = 181,$$

whence $$x = 9, \text{ or } -10.$$

Here either value of x satisfies the *equation*, but the negative root -10 is not applicable to the problem if we regard 9 as one of the "consecutive" numbers. However, if -10 is one of the numbers, $x + 1$, or -9, is the other, and these are consecutive negative numbers.

3. The number of horses that B has is such as to satisfy the equation $2x^2 + 3x = 20$. How many horses has he?

$$2x^2 + 3x = 20,$$

whence $$x = 2\tfrac{1}{2}, \text{ or } -4.$$

The fractional root $2\frac{1}{2}$ satisfies the equation, but the result does not apply to the problem. It is algebraically correct, but practically impossible, since a half horse does not exist.

4. A man bought melons for \$2. Had he bought 10 more for the same money, each would have cost 1 cent less. How many did he buy?

Let x = the number he bought.

Then $\frac{200}{x}$ = the cost of one in cents,

and $\frac{200}{x+10}$ = the cost of one, had he bought 10 more.

$$\therefore \frac{200}{x} - \frac{200}{x+10} = 1, \text{ whence } x = 40, \text{ or } -50.$$

The negative root -50 satisfies the equation, but has no meaning as applied to the problem. It does, however, suggest a similar problem in which "10 more" is replaced by "10 fewer," as follows:

A man bought melons for \$2. Had he bought 10 fewer for the same money, each would have cost 1 cent more.

A solution of this problem gives the results 50 and -40, the latter having no meaning.

5. The sum of two numbers is 60 and their product is 1000. What are the numbers?

Let x and $60 - x$ represent the numbers.

Then $x(60 - x) = 1000,$

or $x^2 - 60x = -1000,$

whence $x = 30 + 10\sqrt{-1}$, or $30 - 10\sqrt{-1}$.

These imaginary roots show that the problem is impossible — that the conditions are inconsistent. It can readily be shown that no two numbers whose sum is 60 can have their product 1000.

6. Divide 50 into two such parts that the sum of their squares shall be 1700.

7. By what number must 18 and 21 each be diminished in order that the product may be diminished by 108?

8. The factors of a product are 25 and 36. By what number must the first be increased and the second diminished in order that the product may be increased by 30?

9. The hypotenuse of a right-angled triangle is 17 inches, and the length of the longer side exceeds that of the shorter by 7 inches. Find the lengths of the sides.

10. A man bought a lot of hogs for $100. If he had bought 5 more for the same money, they would have cost $1 less a head. How many hogs did he buy?

11. What number increased by its square is 110?

12. The sum of the squares of two consecutive numbers is 221. What are the numbers?

13. The sum of two numbers multiplied by the greater is 120. Their difference is 1. What are the numbers?

14. The perimeter of a rectangular field containing 12½ acres is 180 rods. Find the length of the field.

15. What is the price of oranges when 2 more in a dollar's worth lowers the price 1 cent a dozen?

16. The length of a rectangular lot exceeds its width by 15 rods. If it were 10 rods longer and 15 rods wider, its area would be doubled. Find the length of the lot.

17. A bicyclist rode 180 miles at a uniform rate. If he had ridden 3 miles an hour slower than he did, it would have taken him 3 hours longer to ride the same distance. At what rate did he ride?

18. A and B start at the same time to travel to P, which is 36 miles distant. A travels 1 mile an hour faster than B and arrives at P 3 hours sooner. What is the hourly rate of each?

19. A man who traveled 360 miles found that he could have made the trip in an hour less had he traveled 4 miles an hour faster. At what rate did he travel?

20. One of two numbers exceeds 20 by as much as the other is less than 20; their product is 375. Find the numbers.

21. What number exceeds its reciprocal by 4.8?

22. The product of two numbers is 21, and their difference is 4. Find the numbers.

23. One side of a rectangle is 4 inches longer than the other, and the area is 117 square inches. What are the dimensions?

24. The product of two consecutive numbers exceeds the larger by 224. Find the numbers.

25. The product of three consecutive numbers is equal to 8 times the middle number. What are the numbers?

26. The area of a certain square may be quadrupled by increasing each of its sides by 4 rods. Find the side.

27. The diagonal of a rectangular field is 27 rods longer than the longer side and 54 rods longer than the shorter side. Find the area of the field.

28. Find two consecutive odd numbers, the sum of whose squares is 290.

29. A man can row down a stream 16 miles and back in 10 hours. If the stream runs 3 miles an hour, what is his rate of rowing in still water?

30. A merchant bought a number of yards of cloth for $100. He kept 5 yards and sold the rest at $2 a yard more than he gave, and received $20 more than he originally paid. How many yards did he buy?

31. A and B can do a piece of work in 6 days. It takes B 5 days longer than A to do the work alone. In how many days can each do the work alone?

32. A parquet floor 24 feet by 18 feet is made of blocks of wood, each of which is 12 inches longer than broad. If it takes 972 such blocks, how large is each one?

33. Divide a line a foot long into two parts such that the square on one part may be equal to the rectangle contained by the whole and the other part.

34. What is the radius of a circle whose area would be doubled by increasing its radius 2 feet?

35. The fence around a rectangular garden is 128 yards long, and the area of the garden is 1008 square yards. Find its length.

36. An officer wishes to arrange 4000 men in a solid body, so that each rank may exceed each file by 30 men. How many must be placed in rank and file?

37. The perimeter of a room is 56 feet, and the area of the floor equals 48 times the difference between its length and breadth. What is the length of the room?

38. A man paid $240 for a lot of sheep, but because 32 of them died the average cost of the rest was $2 more a head than at first. How many sheep did he buy?

---

39. In a rectangle containing 216 square feet the length exceeds the width by 6 feet. Find the length.

40. There is a certain number whose digits, beginning with ones, are consecutive. The product of the ones and tens, plus the product of the tens and hundreds, equals the sum of the digits plus 20. Find the number.

41. A dealer sold some books for $24, and gained a rate per cent equal to the number of dollars paid for the books. What was the cost of the books?

42. A broker sold a certain number of railroad shares for $3240. A few days later, the price having fallen $9 a share, he buys, for the same sum, 5 more shares than he had sold. How many shares did he sell? At what price did he buy?

43. One leg of a right-angled triangle exceeds the other by 21 meters, and the area of the triangle is 270 square meters. Find the sides.

44. The denominator of a certain fraction exceeds the square of half the numerator by 1, and the product of the sum and difference of numerator and denominator is 225. What is the fraction?

45. A farmer bought a certain number of sheep for $48. If he had bought 4 less for the same money, they would have cost him $1 apiece more. How many did he buy?

46. A cistern is supplied with water by two pipes, A, B. If both pipes are open, the cistern will be filled in 4 hours. The pipe A will fill it in 6 hours less time than pipe B requires. Find the time in which each pipe alone will fill it.

47. A box contains 550 coins of silver and gold. Each silver coin is worth as many cents as there are gold coins, and each gold coin is worth as many cents as there are silver coins, and the whole is worth $500. How many coins are there of each kind?

48. There is a number consisting of two digits whose sum is 12 and the sum of whose squares is 90. What is the number?

49. A and B start at the same time to walk 20 miles. A travels a mile an hour faster than B and arrives an hour earlier. At what rate did each travel?

50. Find three consecutive numbers whose sum is equal to $\frac{3}{7}$ of the product of the last two.

51. The cost of lunch for a party was $15. If each paid 20 cents more than there were persons, how many were in the party?

52. A party had a dinner that cost $50. If there had been 5 persons more, the share of each would have been 50 cents less. How many persons were in the party?

53. Find two consecutive numbers, the sum of whose reciprocals is $\frac{5}{6}$.

54. A man worked a certain number of days for $45. Had he received a dollar a day less than he did, he would have had to work 16 days longer for the same money. How many days did he work?

55. What number is $\frac{5}{6}$ greater than its reciprocal?

56. The frame of a picture 18 inches by 12 inches is of uniform width, and its area is equal to that of the picture. How wide is the frame?

57. If there were 2 fewer telegraph poles in a mile, the distance between the poles would be increased by 24 feet. If the poles are placed at equal intervals, find the number of poles in a mile.

58. A certain number is expressed by two digits whose product is 35, and if 18 is subtracted from the number, the order of digits will be reversed. Find the number.

59. By traveling 4 kilometers an hour faster than usual a train made a run of 675 kilometers in one hour less than its regular time. How many kilometers an hour did it travel?

60. A garden 56 yards long and 16 yards wide is surrounded by a walk of uniform width, without the garden. If the walk contains 640 square yards, what is its width?

61. The difference between two numbers is 9, and the square of the larger minus twice the square of the smaller is 158. Find the numbers.

62. If a bar of iron weighing 40 pounds be drawn out 2 feet longer, it will weigh one pound less per linear foot. What is its length?

63. A man bought two farms for $2800 each. The larger contained 10 acres more than the smaller, but he paid $5 more an acre for the smaller than for the larger. How many acres were in each?

64. A man bought a certain quantity of sugar for $66. If sugar were to rise 1 cent a kilo, he would obtain 50 kilos less for the same money. How many kilos of sugar did he buy?

65. A rectangular lot 20 rods by 15 rods is surrounded by a fence within which is a drive occupying as much area as the rest of the lot. Find the width of the drive.

66. A rectangular lawn 20.5 meters long and 8.5 meters wide is surrounded by a path of uniform width, wholly without the lawn. If the area of the path is 62 square meters, what is its width?

67. If the circumference of the fore wheel of a buggy is 1 foot less than that of the hind wheel, and the former makes 48 more revolutions than the latter in going a mile, what is the circumference of each wheel?

68. A cistern is fitted with two pipes, A and B. A can fill it 3 minutes sooner than B can empty it. If they run together, the cistern will be filled in 18 minutes. In what time will it be filled if A is open and B is closed?

SIMULTANEOUS QUADRATIC EQUATIONS

362. In a system of quadratic equations, as in a linear system, there must be as many equations given as there are unknown numbers, otherwise there can be no definite solution obtained. The given equations must be *consistent* and *independent* (Arts. 221–223).

In simultaneous quadratic equations, as in simple equations, a solution requires the *elimination* of all but one of the unknown numbers; and, generally, by elimination, two simultaneous quadratic equations containing two unknown numbers will produce an equation of the *fourth degree*, which is usually insolvable by means of quadratics. However, a solution may be effected by the methods for solving quadratics in the following special cases:

1. *When one equation is linear.*

2. *When each equation is quadratic and one is homogeneous, or when both are homogeneous* (Art. 106).

3. *When the equations are symmetric with respect to the unknown numbers* — when the unknown numbers can be interchanged without affecting the equations (Art. 107).

Certain simultaneous equations that do not fall under the preceding classes may be readily solved by special devices. Often equations that belong to one or more of these classes can be solved more expeditiously by means of such devices.

363. It will tend to simplicity to consider first a particular problem in which each of the equations contains only the squares of the unknown numbers. Obviously in this case one of the unknown numbers may be eliminated by addition or subtraction, and the value of the other be found by substitution.

Solve the system

$$\begin{cases} 5x^2 - 3y^2 = 33. & (1) \\ 3x^2 + 2y^2 = 35. & (2) \end{cases}$$

Multiplying (1) by 2, $10x^2 - 6y^2 = 66.$ (3)

Multiplying (2) by 3, $9x^2 + 6y^2 = 105.$ (4)

Adding (3) and (4), $19x^2 = 171.$

$\therefore x = \pm 3.$

Substituting either of these values of x in (1), we find $y = \pm 2$. That is, x and y each has two values. When $x = 3$, $y = \pm 2$; when $x = -3$, $y = \pm 2$. Hence the following pairs of values satisfy both equations:

$$\begin{cases} x = 3, \\ y = 2; \end{cases} \quad \begin{cases} x = 3, \\ y = -2; \end{cases} \quad \begin{cases} x = -3, \\ y = 2; \end{cases} \quad \begin{cases} x = -3, \\ y = -2. \end{cases}$$

Are the given equations homogeneous with respect to x and y? Verify these results by substituting the pairs of values in the given equation.

364. One Equation Linear.

In this case, one of the unknown numbers can be found in terms of the other from the linear equation, and that value substituted in the quadratic. The resulting equation will be a quadratic in one unknown number, which may always be solved.

Certain special examples in which one equation is of the third degree and the other of the first may be solved in a similar manner.

1. Solve the system

$$\begin{cases} 5x^2 - 3xy = 27. & (1) \\ x + y = 5. & (2) \end{cases}$$

From (2) we find $y = 5 - x$.

Substituting $5 - x$ for y in (1), we have

$$5x^2 - 3x(5 - x) = 27,$$

or $$8x^2 - 15x = 27. \quad (3)$$

Solving this equation, we obtain $x = 3$, or $-\frac{9}{8}$.

Substituting 3 for x in (2), $y = 2$.

Substituting $-\frac{9}{8}$ for x in (2), $y = 6\frac{1}{8}$.

$$\therefore \begin{cases} x = 3, \\ y = 2; \end{cases} \qquad \begin{cases} x = -\frac{9}{8}, \\ y = 6\frac{1}{8}. \end{cases}$$

Note that the equations of the given system are simultaneous, but they are not simultaneous *quadratics*, since one of the equations is of the first degree. In this case we should substitute in the linear equation the values found for one unknown number in order to obtain the values of the other. This is important because the quadratic equation and the equation obtained by substituting in it the value of one unknown number derived from the linear equation are not necessarily equivalent to the given system. In this example, the system (1), (3) is not necessarily equivalent to the given system (1), (2). How many pairs of roots result from this solution? How many pairs should be expected when both equations are quadratic?

2. Solve the system

$$\begin{cases} x^3 - y^3 = 61. & (1) \\ x - y = 1. & (2) \end{cases}$$

From (2), $y = x - 1.$

Substituting this value in (1),

$$x^3 - (x-1)^3 = 61,$$

or $$x^3 - x^3 + 3x^2 - 3x + 1 = 61,$$

or $$x^2 - x - 20 = 0.$$

$$\therefore (x-5)(x+4) = 0.$$

$$\therefore x = 5, \text{ or } -4;$$

$$y = 4, \text{ or } -5.$$

Solve the following systems:

3. $x + y = 5.$
$x^2 - 2y^2 = 1.$

4. $3x + 4y = 18.$
$5x^2 - 3xy = 2.$

5. $x - y = 1.$
$xy = 12.$

6. $x + y = 9.$
$xy = 20.$

7. $\frac{1}{x} + \frac{1}{y} = 2.$
$x + y = 2.$

8. $x^2 + xy = 12.$
$x - y = 2.$

9. $x^2 + xy + y^2 = 63.$
$x - y = -3.$

10. $x^2 + y^2 = 34.$
$x - y = 2.$

11. $x^2 - xy = 3.$
$x + 2y = 7.$

12. $2x + y = 3.$
$x^2 + xy - y^2 = 1.$

13. $x^2 - 2xy - y^2 = 1.$
$x + y = 2.$

14. $\frac{x}{3} + \frac{y}{4} = 5.$
$\frac{3}{x} + \frac{4}{y} = \frac{5}{6}.$

15. $x^3 - y^3 = 19.$
$x - y = 1.$

16. $x^3 + 27y^3 = 98.$
$x + 3y = 2.$

365. Homogeneous Equations.

The solution of a pair of simultaneous quadratic equations is always possible when one of the equations is homogeneous, or when both are homogeneous except for the constant terms.

1. Solve the system

$$\begin{cases} 3x^2 + 2xy - y^2 = 0. & (1) \\ x^2 + y^2 + 2x = 12. & (2) \end{cases}$$

Dividing (1) by y^2, $3\left(\frac{x}{y}\right)^2 + 2\left(\frac{x}{y}\right) - 1 = 0$, a quadratic in $\frac{x}{y}$.

By factoring, $\left(\frac{3x}{y} - 1\right)\left(\frac{x}{y} + 1\right) = 0.$

$$\therefore \frac{x}{y} = \frac{1}{3}, \text{ or } -1,$$

and $x = \frac{1}{3}y$, or $-y$.

Substituting $\frac{1}{3}y$ for x in (2),

$$\frac{y^2}{9} + y^2 + \frac{2y}{3} = 12.$$

$$\therefore 5y^2 + 3y = 54.$$

Solving, $y = 3$, or $-\frac{18}{5}$.

And since $x = \frac{1}{3}y$, $x = 1$, or $-\frac{6}{5}$.

Substituting $-y$ for x in (2),

$$y^2 + y^2 - 2y = 12,$$

or $y^2 - y = 6.$

$$\therefore (y - 3)(y + 2) = 0.$$

$$\therefore y = 3, \text{ or } -2.$$

And since $x = -y$, $x = -3$, or 2.

$$\therefore x = 1, -\frac{6}{5}, -3, 2;$$
$$y = 3, -\frac{18}{5}, 3, -2.$$

Note that only one of the two given equations is homogeneous.

2. Solve the system

$$\begin{cases} x^2 - xy + y^2 = 21. & (1) \\ x^2 + 2y^2 = 27. & (2) \end{cases}$$

Multiplying (1) by 9, $9x^2 - 9xy + 9y^2 = 189.$ (3)

Multiplying (2) by 7, $7x^2 + 14y^2 = 189.$ (4)

Subtracting (4) from (3), $2x^2 - 9xy - 5y^2 = 0.$ (5)

Factoring, $(x - 5y)(2x + y) = 0.$

$\therefore x = 5y$, or $-\frac{1}{2}y.$

Substituting $5y$ for x in (2), $25y^2 + 2y^2 = 27,$

or $y^2 = 1.$

$\therefore y = \pm 1.$

Substituting $-\frac{y}{2}$ for x in (2), $\frac{y^2}{4} + 2y^2 = 27,$

or $y^2 = 12.$

$\therefore y = \pm\sqrt{12} = \pm 2\sqrt{3}.$

Since $x = 5y,$ $x = \pm 5,$

and since $x = -\frac{1}{2}y,$ $x = \mp\sqrt{3}.$

If (5) had not been easily factored, we should have divided both members by y^2, as in the preceding solution.

Since in equations of this class we seek the value of $\frac{x}{y}$, we can often more conveniently solve by letting $\frac{x}{y} = v$, or $x = vy$, as follows:

Let $x = vy$, in the example just solved.

Substituting in (1), $v^2y^2 - vy^2 + y^2 = 21.$ (3)

Substituting in (2), $v^2y^2 + 2y^2 = 27.$ (4)

From (3), $y^2 = \frac{21}{v^2 - v + 1}.$ (5)

From (4), $y^2 = \frac{27}{v^2 + 2}.$ (6)

Equating values of y^2, $\frac{21}{v^2 - v + 1} = \frac{27}{v^2 + 2}.$ (7)

Clearing, etc., $2v^2 - 9v = 5.$ (8)

Solving (8), we find $v = 5,$ or $-\frac{1}{2}.$

Putting 5 for v in (5) or (6), $y = \pm 1.$

Then $x = vy = 5y = \pm 5.$

Putting $-\frac{1}{2}$ for v in (5) or (6), $y = \pm\sqrt{12} = \pm 2\sqrt{3}.$

Then $x = vy = -\frac{1}{2}y = \mp\frac{1}{2}\sqrt{12} = \mp\sqrt{3}.$

Note that each of the given equations is a quadratic and homogeneous. It is only in equations of this kind that the special expedient of putting $x = vy$ is of advantage.

Solve the following systems:

3. $x^2 - xy = 2.$
$2x^2 + y^2 = 9.$

4. $x^2 + 3xy = 28.$
$xy + 4y^2 = 8.$

5. $x^2 - 2xy = 15.$
$3xy - y^2 = 14.$

6. $xy + x^2 = 15.$
$xy - y^2 = 2.$

7. $x^2 + y^2 = 61.$
$x^2 - xy = 6.$

8. $3x^2 + y^2 = 12.$
$5xy - 4x^2 = 11.$

9. $4x^2 + 5y^2 = 216.$
$(2x - y)^2 = 144.$

10. $3xy - 4y^2 = 2.$
$x^2 + 3xy - 5y^2 = 7.$

11. $x^2 + xy + y^2 = 63.$
$x^2 - y^2 = -27.$

12. $x^2 + xy + 2y^2 = 74.$
$24y^2 - 25xy = 25x^2.$

13. $x^2 - xy = 0.$
$x^2 + 3xy + 5y^2 = 36.$

14. $x^2 - xy + y^2 = 21.$
$y^2 - 2xy = -15.$

15. $x^2 + 3xy - 2y^2 = 2.$
$2x^2 - 5xy + 6y^2 = 3.$

16. $2x^2 - 2x + y = 13.$
$x^2 - 4xy + 3y^2 = 0.$

17. $x^2 + 2y^2 = 22.$
$3y^2 - xy - x^2 = 17.$

18. $5x^2 - 2xy - 7y^2 = 0.$
$xy - y^2 = 40.$

366. Symmetrical Equations.

The solution of a system of two simultaneous equations which are symmetric with respect to the two unknown numbers may be effected by combining them in such a way as to first obtain values of $x + y$ and $x - y$, and then the values of x and y themselves. Sets of equations that are symmetric except with respect to the signs of the terms can often be solved by this method.

It is often more convenient to adopt the special device of substituting for the unknown numbers the sum and difference

of two other numbers, as $x = u + v$, and $y = u - v$, and first solve for u and v.

Many symmetric equations of higher degree than the second yield to this method of solution.

1. Solve the system

$$\begin{cases} x + y = 7. & (1) \\ xy = 12. & (2) \end{cases}$$

Squaring (1), $x^2 + 2xy + y^2 = 49.$ (3)

Subtracting 4 times (2) from (3),

$$x^2 - 2xy + y^2 = 1.$$

Extracting square root, $x - y = \pm 1.$ (4)

Adding (4) and (1), $2x = 8,$ or $6.$

$$\therefore x = 4, \text{ or } 3.$$

$$\therefore y = 3, \text{ or } 4, \text{ by symmetry.}$$

2. Solve the system

$$\begin{cases} x^2 + y^2 = 65. & (1) \\ x - y = -3. & (2) \end{cases}$$

(*a*)

Multiplying (1) by 2, $2x^2 + 2y^2 = 130.$ (3)

Squaring (2), $x^2 - 2xy + y^2 = 9.$ (4)

Subtracting (4) from (3), $x^2 + 2xy + y^2 = 121.$

Extracting square root, $x + y = \pm 11.$ (5)

Adding (5) and (2), $2x = 8,$ or $-14.$

$$\therefore x = 4, \text{ or } -7.$$

Subtracting (5) from (2), $-2y = -14,$ or $8.$

$$\therefore y = 7, \text{ or } -4.$$

(*b*)

Let $x = u + v,$ and $y = u - v.$

Then (1) becomes, $u^2 + 2uv + v^2 + u^2 - 2uv + v^2 = 65,$

or $2u^2 + 2v^2 = 65.$ (3)

Also, (2) becomes, $2v = -3.$ (4)

$$\therefore v = -\tfrac{3}{2}.$$

Substituting $-\frac{3}{2}$ for v in (3), $2u^2 + \frac{9}{2} = 65.$

$$\therefore u = \pm \tfrac{11}{2}.$$

$$\therefore x = u + v = \pm \tfrac{11}{2} + (-\tfrac{3}{2}) = 4, \text{ or } -7.$$

$$\therefore y = u - v = \pm \tfrac{11}{2} - (-\tfrac{3}{2}) = 7, \text{ or } -4.$$

3. Solve the system

$$\begin{cases} x^3 + y^3 = 35. & (1) \\ x + y = 5. & (2) \end{cases}$$

Let $x = u + v$, and $y = u - v$.

Then (1) becomes, $2u^3 + 6uv^2 = 35$, (3)

and (2) becomes, $u = \frac{5}{2}$. (4)

Substituting $\frac{5}{2}$ for u in (3), $\frac{125}{4} + 15v^2 = 35$.

$\therefore v = \pm \frac{1}{2}$.

$x = u + v = \frac{5}{2} \pm \frac{1}{2} = 3$, or 2.

$\therefore y = 2$, or 3, by symmetry.

As an exercise have the student find $x - y$ by dividing the members of (1) by the members of (2), etc., and then proceed to find x and y.

4. Solve the system

$$\begin{cases} x^4 + y^4 = 82. & (1) \\ x + y = 2. & (2) \end{cases}$$

Raising (2) to the 4th power,

$$x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4 = 16. \quad (3)$$

Subtracting (1) from (3),

$$4x^3y + 6x^2y^2 + 4xy^3 = -66. \quad (4)$$

$4xy \times$ square of (2), $4x^3y + 8x^2y^2 + 4xy^3 = 16xy$. (5)

Taking (4) from (5), $2x^2y^2 = 16xy + 66$,

from which we find $xy = -3$, or 11.

We now have two systems of equations to solve,

$$\begin{cases} x + y = 2, \\ xy = -3, \end{cases} \text{ and } \begin{cases} x + y = 2, \\ xy = 11. \end{cases}$$

Solving the first, $\begin{matrix} x = 3, \\ y = -1, \end{matrix}$ or $\begin{matrix} x = -1, \\ y = 3. \end{matrix}$

Solving the second, $\begin{matrix} x = 1 + \sqrt{-10}, \\ y = 1 - \sqrt{-10}, \end{matrix}$ or $\begin{matrix} x = 1 - \sqrt{-10}, \\ y = 1 + \sqrt{-10}. \end{matrix}$

Let the student solve by letting $x = u + v$ and $y = u - v$ for purposes of comparison. The first method of solution seeks to combine the equations in such manner as to cause the highest powers of x and y to disappear. What is the effect of the second method?

Solve the following systems:

5. $x+y=15.$
$xy=36.$

6. $x^2+y^2=74.$
$x+y=12.$

7. $x^2+y^2=29.$
$x-y=3.$

8. $x^2+y^2=74.$
$xy=35.$

9. $x^2+xy+y^2=61.$
$x+y=9.$

10. $x^2+y^2=13.$
$x+y+xy=11.$

11. $\frac{1}{x^3}-\frac{1}{y^3}=19.$
$\frac{1}{x}-\frac{1}{y}=1.$

12. $4(x+y)=3xy.$
$x^2+y^2+x+y=26.$

13. $x^3+y^3=91.$
$x^2-xy+y^2=13.$

14. $4(x+y)+3xy=0.$
$x^2+x+y+y^2=20.$

15. $x^4+y^4=97.$
$x+y=1.$

16. $\frac{1}{x}+\frac{1}{y}=7.$
$\frac{1}{x^2}+\frac{1}{y^2}=25.$

17. $x^3-y^3=61.$
$x-y=1.$

18. $x^2+y^2+3(x+y)=4.$
$3x^2+4xy+3y^2=3.$

367. In a system of two equations, each having 0 as its second member, if the first member of one or of both can be resolved into rational factors, the given system is equivalent to as many derived systems as there are factors in both equations.

Thus, the system $\begin{cases}(x-a)(y-b)=0,\\ x^2+y^2-ab=0,\end{cases}$ is equivalent to the two derived systems

$$\begin{cases}x-a=0,\\ x^2+y^2=ab,\end{cases} \text{ and } \begin{cases}y-b=0,\\ x^2+y^2=ab;\end{cases}$$

and the system $\begin{cases}(x-m)(y+n)=0,\\ (x-a)(y-b)=0,\end{cases}$ is equivalent to the four derived systems

$$\begin{cases}x-m=0,\\ x-a=0;\end{cases} \quad \begin{cases}x-m=0,\\ y-b=0;\end{cases} \quad \begin{cases}y+n=0,\\ x-a=0;\end{cases} \quad \begin{cases}y+n=0,\\ y-b=0.\end{cases}$$

The solutions of these derived systems are the solutions of the given system.

It follows from the preceding that *certain systems of quadratic equations can be solved by the method of factoring.*

1. Solve the system

$$\begin{cases} (x-3)(y-5)=0. \\ x+y=9. \end{cases}$$

The derived systems are

$$\begin{cases} x-3=0, \\ x+y=9, \end{cases} \text{ and } \begin{cases} y-5=0, \\ x+y=9. \end{cases}$$

The solution of the first is $x=3,\ y=6$; the solution of the second is $x=4,\ y=5$.

2. Solve the system

$$\begin{cases} 4x^2-8xy+3y^2=0. \\ x^2-y^2-x+5y=6. \end{cases}$$

Factoring the equations, we have

$$(2x-y)(2x-3y)=0,$$
$$(x-y+2)(x+y-3)=0,$$

which is equivalent to the four derived systems:

(1) $\begin{cases} 2x-y=0, \\ x-y+2=0; \end{cases}$ (2) $\begin{cases} 2x-y=0, \\ x+y-3=0; \end{cases}$ (3) $\begin{cases} 2x-3y=0, \\ x-y+2=0; \end{cases}$ (4) $\begin{cases} 2x-3y=0, \\ x+y-3=0. \end{cases}$

From (1) we find $x=2,\ y=4$; from (2), $x=1,\ y=2$; from (3), $x=-6,\ y=-4$; from (4), $x=\frac{6}{5},\ y=\frac{9}{5}$. These are the solutions of the given system.

Solve the following systems:

3. $(x-4)(y+5)=0.$
$(x+2)(y-3)=0.$

4. $(x-2)(y-4)=0.$
$(y-3)(x-5)=0.$

5. $x(x+y)=15.$
$y(y-x)=-2.$

6. $x^2-xy=12.$
$x^2+xy+y^2=76.$

7. $x^2-xy+y^2=3x.$
$x^2+xy+y^2=7x.$

8. $x^2+2y^2=22.$
$xy+4y^2=30.$

9. $x^2-xy+y^2=19.$
$y^2-2xy=-21.$

10. $(x+y)(x-y+1)=0.$
$(x+2)(y+3)=0.$

368. Special Devices.

Many systems of quadratic equations are most easily solved by the use of some special device or expedient. Even equations of higher degree than the second can often be solved by the methods and expedients used in the solution of quadratics. In such cases the success of the student will depend largely upon his ingenuity.

1. Solve the system

$$\begin{cases} x + y = x^2. & (1) \\ 3y - x = y^2. & (2) \end{cases}$$

Subtracting (1) from (2), $2(y - x) = (y - x)(y + x)$.

Dividing by $y - x$, $2 = y + x$.

Putting 2 for $x + y$ in (1), $x^2 = 2$.

$\therefore x = \pm\sqrt{2}$, and $y = 2 \mp \sqrt{2}$.

2. Solve the system

$$\begin{cases} x + y = 10. & (1) \\ x\sqrt{y} = 12. & (2) \end{cases}$$

From (1), $x = 10 - y$.

Then (2) becomes $(10 - y)\sqrt{y} = 12$,

or $10\sqrt{y} - y\sqrt{y} = 12.$ (3)

Putting m for $\sqrt{y}$, we may write (3)

$$10m - m^3 = 12,$$

or $$m^3 - 10m + 12 = 0. \quad (4)$$

Multiplying (4) by m, we have

$$m^4 - 10m^2 + 12m = 0.$$

Adding $4m^2 - 12m + 9$ to each side,

$$m^4 - 6m^2 + 9 = 4m^2 - 12m + 9.$$

$$\therefore m^2 - 3 = \pm(2m - 3).$$

Solving, $m = 2$, or $-1 \pm \sqrt{7}$.

$\therefore \sqrt{y} = 2$, or $-1 \pm \sqrt{7}$.

$\therefore \quad y = 4$, or $8 \mp 2\sqrt{7}$.

Let the student find the values of x.

3. Solve

$$\begin{cases} \sqrt{\dfrac{3x}{x+y}} + \sqrt{\dfrac{x+y}{3x}} = 2. & (1) \\ xy - (x+y) = 54. & (2) \end{cases}$$

Squaring, etc., $$\frac{3x}{x+y} + \frac{x+y}{3x} = 2.$$

Putting $x + y = s$, and clearing, we have

$$9x^2 - 6xs = -s^2,$$

or $$9x^2 - 6xs + s^2 = 0.$$

$$\therefore 3x - s = 0, \text{ and } 3x = s.$$

$$\therefore \quad x + y = 3x, \text{ or } y = 2x.$$

Substituting in (2), $2x^2 - 3x = 54$.

Let the student complete the solution.

4. Solve the system

$$\begin{cases} xy(x+y) = 70. & (1) \\ x^3 + y^3 = 133. & (2) \end{cases}$$

Let $$x = m + n, \; y = m - n.$$

Then (1) and (2) become

$$2m^3 - 2mn^2 = 70. \quad (3)$$

$$2m^3 + 6mn^2 = 133. \quad (4)$$

Adding $3 \times (3)$ to (4), $$8m^3 = 343.$$

Let the student find the values of m, n, x, and y.

5. Solve the system

$$\begin{cases} x^4 + x^2y^2 + y^4 = 133. & (1) \\ x^2 + xy + y^2 = 19. & (2) \end{cases}$$

Factoring (1), $(x^2 + xy + y^2)(x^2 - xy + y^2) = 133.$ (Art. 148)

$$\therefore 19(x^2 - xy + y^2) = 133.$$

$$x^2 - xy + y^2 = 7. \quad (3)$$

Subtracting from (2), $2xy = 12$, whence $xy = 6$.

Adding to (2) and subtracting from (3),

$$x^2 + 2xy + y^2 = 25, \text{ and } x^2 - 2xy + y^2 = 1.$$

$$\therefore x + y = \pm 5, \text{ and } x - y = \pm 1.$$

$$\therefore x = 3, -3, 2, -2, \text{ and } y = 2, -2, 3, -3.$$

6. Solve the system

$$\begin{cases} x^5 + y^5 = 32. & (1) \\ x + y = 2. & (2) \end{cases}$$

Let $x = u + v$, and $y = u - v$.

(2) becomes $u + v + u - v = 2$, or $2u = 2$.

$\therefore u = 1$.

$\therefore x^5 = (1 + v)^5$, and $y^5 = (1 - v)^5$,

and (1) becomes, $(1 + v)^5 + (1 - v)^5 = 32$.

Simplifying, $10v^4 + 20v^2 + 2 = 32$,

or $v^4 + 2v^2 - 3 = 0$.

$\therefore (v^2 - 1)(v^2 + 3) = 0$.

$\therefore v = \pm 1$, or $\pm\sqrt{-3}$.

$\therefore x = u + v = 1 \pm 1$, or $1 \pm \sqrt{-3}$.

$y = u - v = 1 \mp 1$, or $1 \mp \sqrt{-3}$.

$\therefore x = 2,\ 0,\ 1 \pm \sqrt{-3}$.

$y = 0,\ 2,\ 1 \mp \sqrt{-3}$.

7. Solve the system

$$\begin{cases} \dfrac{x+y}{x-y} + \dfrac{x-y}{x+y} = \dfrac{10}{3}. & (1) \\ x^2 + xy = 6. & (2) \end{cases}$$

Let $\dfrac{x+y}{x-y} = z$; then $z + \dfrac{1}{z} = \dfrac{10}{3}$.

$\therefore z^2 - \frac{10}{3}z = -1$.

Completing square, $z^2 - \frac{10}{3}z + (\frac{5}{3})^2 = -1 + \frac{25}{9} = \frac{16}{9}$.

$z - \frac{5}{3} = \pm\frac{4}{3}$.

$\therefore z = 3$, or $\frac{1}{3}$.

But $z = \dfrac{x+y}{x-y} = 3$, or $\frac{1}{3}$.

$\therefore x = \pm 2y$.

This value in (2) gives $(\pm 2y)^2 + y(\pm 2y) = 6$,

or $4y^2 \pm 2y^2 = 6$.

$6y^2 = 6$, or $2y^2 = 6$.

$\therefore y = \pm 1$, or $\pm\sqrt{3}$.

$x = \pm 2,$ or $\mp 2\sqrt{3}$.

Solve the following:

8. $xy + 1 = x + y.$
$xy = 2.$

9. $x^3 + y^3 = 72.$
$x + y = 6.$

10. $x^3 + y^3 = 18\,xy.$
$x + y = 12.$

11. $x - y = 4.$
$(x + y)^2 + (x + y) = 20.$

12. $x^4 + y^4 = 272.$
$x + y = 6.$

13. $\sqrt[3]{x} - \sqrt[3]{y} = 1.$
$x - y = 19.$

14. $2\,x + y = x^2.$
$2\,y + x = y^2.$

15. $x = m\sqrt{x + y}.$
$y = n\sqrt{x + y}.$

369. In solving a system of simultaneous equations, the student must for the most part decide for himself which method of elimination it is best to employ, and what special devices can be used to advantage. The following general directions which summarize the methods employed in the preceding exercises may be found of some value in attacking the sets of equations in the following exercises:

I. If *one* of the equations is *linear*, it is usually best to find the value of one unknown number in terms of the other from the linear equation, and then substitute that value in the quadratic. The resulting equation is solved as a quadratic.

II. When *one* of the equations is *homogeneous*, or when *both are homogeneous except for the constant terms*, one of the equations can be reduced to the form of a quadratic in $\frac{x}{y}$, and after $\frac{x}{y}$ has been found, x will be known as some multiple of y, and this value can be substituted in the other equation.

It is often better in practice to let $x = vy$, and then find v.

III. When the equations are *symmetric*, it is usually better to put $x = u + v$ and $y = u - v$, and first solve for u and v.

It is often preferable to so combine the equations as to first obtain the values of $x + y$ and $x - y$ directly, and then find x and y themselves. This is simpler when $x + y$ and $x - y$ are easily found.

IV. When the equations can be *readily factored* without combining, it is often advisable to solve by the method of factoring used in Art. 367.

V. It is not practicable to classify the *special devices* that apply in the solution of different systems. The student must study each example carefully to discover the device that applies in that particular case. The illustrative solutions in Art. 368 will suggest some of the devices that are most often applicable.

The preceding directions are given as *hints* to the student, and are not intended to be followed in all cases. Many systems falling under the more general cases may be more expeditiously solved by special methods. In attacking systems that are best solved thus, the answers to questions like the following will usually lead the student to discover the method or device that is best adapted: Can one equation be divided by the other, member by member? Will raising one equation to the same power (degree) as the other enable us to eliminate the higher powers? If a multiple of one equation is added to or subtracted from the other, will the members of the resulting equation be perfect squares? If we let some number, as z, represent the sum, difference, product, or quotient of the two unknown numbers, can we readily find the value of z? With practice other devices will suggest themselves to the student.

Solve the following systems of equations by appropriate methods, and check the results:

1. $x + y = 4.$
   $xy = 3.$
2. $x - y = 3.$
   $xy = 10.$
3. $x + 5y = 21.$
   $xy = 4.$
4. $2x - 3y = 0.$
   $xy = 6.$
5. $x - 2y = 0.$
   $x^2 + y^2 = 20.$
6. $x^2 + xy = 15.$
   $x - y = 1.$
7. $x(3x - 2y) = 5.$
   $x - y = 2.$
8. $3x + 2y = 11.$
   $x(x + y) = 12.$
9. $3x^2 - y^2 = 2.$
   $2x + y = 11.$
10. $x^2 + y^2 = 13.$
    $x(x + y) = 10.$

11. $5(x+y)=xy.$
$xy=180.$

12. $\frac{1}{x}+\frac{1}{y}=6.$
$\frac{1}{x^2}+\frac{1}{y^2}=20.$

13. $x^2-2xy+5=0.$
$(x-y)^2-4=0.$

14. $xy+y^2=4x+y.$
$5xy+2y^2=8x+5y.$

15. $x^2+y^2+1=3xy.$
$2(xy+4)=3y^2.$

16. $3x^2-5xy=7.$
$4xy-2x^2=14.$

17. $x+y=13.$
$\sqrt{x}+\sqrt{y}=5.$

18. $x^3+x=9y.$
$x^2+1=6y.$

19. $x^2+2xy-3y^2=5.$
$2x^2+5y^2=13.$

20. $4x^2-xy=50.$
$5x-2y=5.$

21. $3x^2-2y=6.$
$x^2+3y=13.$

22. $5x^2+3y^2=48.$
$4x^2-y^2=35.$

23. $x^2+y^2=40.$
$x-y=4.$

24. $x^2+y^2=25.$
$x+y=7.$

25. $x^2+y^2=10.$
$x+3y=10.$

26. $xy(x-y)=6.$
$x^3-y^3=19.$

27. $x(2x-3y)+y^2=14.$
$2x-y-7=0.$

28. $3x-2xy+2y=0.$
$xy=6.$

29. $x^2-2xy=5.$
$x^2-y^2=21.$

30. $x^3+y^3=126.$
$x^2-xy+y^2=21.$

31. $\frac{1}{x^3}+\frac{1}{y^3}=133.$
$\frac{1}{x}+\frac{1}{y}=7.$

32. $x^2y-xy^2=20.$
$x^3-y^3=61.$

33. $x^2+4xy+4y^2=16.$
$3x^2+8y^2=20.$

34. $x+y=35.$
$x^{\frac{1}{3}}+y^{\frac{1}{3}}=5.$

35. $x^2-xy=a^2+b^2.$
$xy-y^2=2ab.$

36. $y^2+ay+bx=0.$
$x-y=a.$

37. $x+y+\sqrt{x+y}=6.$
$x^2+y^2=10.$

38. $ax^2+b(x^2+y^2)=m.$
$cy^2+d(x^2+y^2)=n.$

39. $\dfrac{x+y}{x-y}+\dfrac{x-y}{x+y}=\dfrac{5}{2}.$
$x^2+xy=12.$

40. $x+y+\sqrt{x+y}=12.$
$x-y+\sqrt{x-y}=2.$

41. $x^4+3x^2y^2=28.$
$x^2y^2+4y^4=8.$

42. $x^2y^2-6x=34-3y.$
$3xy+y=18+2x.$

43. $x^2+2xy+x+3y=75.$
$y^2+x-y=24.$

44. $x^4+x^2y^2+y^4=91.$
$(x^2-xy+y^2)(x-y)^2=28.$

45. $xy(x+y)=30.$
$x^3+y^3=35.$

46. $x^2y+xy^2=180.$
$x^2y^2=400.$

47. $x^2+xy+y^2=84.$
$x-\sqrt{xy}+y=6.$

48. $x^2+xy-6y^2=21.$
$xy-2y^2=4.$

49. $x^5-y^5=211.$
$x-y=1.$

50. $(3x-2y)(2x-3y)=26.$
$x-2y+1=0.$

51. $x^3-y^3=6a^2b+2b^3.$
$xy(x-y)=2b(a^2-b^2).$

SIMULTANEOUS QUADRATICS WITH THREE UNKNOWNS

370. In general, three simultaneous quadratic equations containing three unknown numbers do not yield to the methods for solving quadratics. However, many special cases can be solved by these methods, and a few illustrative problems and solutions are here given.

1. Solve the system

$$\begin{cases} xy=24. & (1)\\ xz=32. & (2)\\ yz=12. & (3)\end{cases}$$

(1) × (2), $\quad xy\times xz=24\times 32.$ (4)

(4) ÷ (3), $\quad x^2=64.$ (5)

$\therefore\ x=\pm 8,\ y=\pm 3,$ and $z=\pm 4.$

2. Solve the system

$$\begin{cases} x(x+y+z)=18. & (1)\\ y(x+y+z)=12. & (2)\\ z(x+y+z)=6. & (3)\end{cases}$$

Adding (1), (2), and (3), $(x+y+z)(x+y+z)=36.$

$$\therefore (x+y+z)^2=36.$$

$$\therefore x+y+z=\pm 6.$$

Substituting this value in (1), (2), and (3), respectively,

$$x=\pm 3.$$
$$y=\pm 2.$$
$$z=\pm 1.$$

3. Solve the system

$$\begin{cases} x+y+z=1. & (1)\\ 3x-2y+2z=2. & (2)\\ x^2+2y^2+x+z=1. & (3)\end{cases}$$

Eliminating z from (1) and (2), $y=\frac{1}{4}x.$

Eliminating y from (1) and (2), $5x+4z=4,$

whence $$z=\frac{4-5x}{4}.$$

Using these values in (3), we have

$$x^2+\tfrac{1}{8}x^2+x+\frac{4-5x}{4}=1.$$

Clearing, etc., $$9x^2-2x=0.$$

$$x(9x-2)=0.$$

$$\therefore x=0, \text{ or } \tfrac{2}{9}.$$

Hence $$y=0, \text{ or } \tfrac{1}{18},$$
$$z=1, \text{ or } \tfrac{13}{18}.$$

Solve the following systems:

4. $x+y-2z=-9.$
$3x+2y+z=9.$
$x^2+y^2+z^2=30.$

5. $xy+xz=27.$
$yz+yx=32.$
$zx+zy=35.$

6. $x-y=1.$
$2x+3y+z=8.$
$x^2+y+z=6.$

7. $2xy+x+z=22.$
$2yz+y+z=58.$
$2xz+x+z=32.$

8. $x^2+3xy-3xz=9.$
$2x+3y=9.$
$3x+4z=13.$

9. $x(x+y+z)=a^2.$
$y(x+y+z)=b^2.$
$z(x+y+z)=c^2.$

10. $2x+y-z=3.$
$3x+y+2z=7.$
$xy+y^2+z=7.$

11. $yz+2(y+z)=11.$
$zx+2(x+z)=8.$
$xy+2(x+y)=16.$

PROBLEMS

371. 1. Find two numbers whose sum is 15 and whose product is 44.

2. The difference of two numbers is 4, and their product is 117. What are the numbers?

3. Find two numbers whose sum is 19, and the sum of whose squares is 185.

4. It takes 84 rods of fence to inclose a rectangular lot containing 2 acres. How long is the lot?

5. The difference of two numbers multiplied by the greater is 120, and multiplied by the less is 56. What are the numbers?

6. The product of two numbers is 24, and their sum added to the sum of their squares is 62. Find the numbers.

7. The area of a rectangle is 96 square inches. If the length and width are each increased by 2 inches, the area will be 144 square inches. Find the dimensions of the rectangle.

8. The sum of two numbers is 65, and the sum of their square roots is 9. What are the numbers?

9. Find two numbers whose difference multiplied by the difference of their squares is 32, and whose sum multiplied by the sum of their squares is 272.

10. If a rectangular field were 75 rods longer and 20 rods wider, its length would be double its width, and its area would be double what it is now. What are the dimensions of the field?

11. The diagonal of a rectangle is 20 rods. If the rectangle were 4 rods shorter and 4 rods wider, the diagonal would still be 20 rods. What is the area of the rectangle?

12. The product of two numbers is 31 greater than twice their sum and 67 less than the sum of their squares. Find the numbers.

13. A rectangular field containing $6\frac{3}{4}$ acres was doubled in size by increasing its length 18 rods and its width 10 rods. What was its former length?

14. I can buy 6 pairs of gloves and 2 pairs of shoes for \$15. For \$14 I can buy 4 more pairs of gloves than I can buy of shoes for \$9. What is the price of a pair of each?

15. The area of a rectangle is 168 square meters, and the length of its diagonal is 25 meters. Find its dimensions.

16. Find two numbers such that the first increased by twice the second is 24, and the sum of their squares is 149.

17. The floor of a room contains 210 square feet, each of the two side walls 135 square feet, and each of the two end walls 126 square feet. Find the dimensions.

18. The combined area of two square fields, A, B, is 1300 square rods, and 200 rods of fence are required to inclose both. What is the value of each field at \$2.25 a square rod?

19. The difference of two numbers is 4, and the difference of their cubes is 316. What are the numbers?

20. A and B bought 600 acres of land for \$600, each paying \$300. In dividing the land, A took the best land and paid \$$\frac{3}{4}$ more an acre than B. How many acres did each get, and at what price?

21. The sum of two numbers is equal to the difference of their squares, and the product of the numbers exceeds 4 times their sum by 4. Find the numbers.

22. The difference of two numbers is 3, and the difference of their cubes exceeds the difference of their squares by 96. What are the numbers?

23. A man paid \$ 285 for two horses. He sold one for \$ 150 and the other for \$ 128, thereby gaining as much per cent on the first as he lost on the second. What was the cost of each?

24. Two men can do a piece of work in $13\frac{1}{3}$ days. How long will it take each of them to do the work, if it takes one 6 days longer than the other?

25. A man found that he could buy 6 more calves than cows for \$ 100, and that the cost of 4 cows was \$ 300 less than the cost of 10 times as many calves. What was the price of each?

26. A number is expressed by two digits. The sum of the squares of the digits is 40. If 6 times the tens' digit be subtracted from the number, the order of the digits will be reversed. Find the number.

27. The sum of the numerator and denominator of a fraction is 7. If each be increased by 3, the value of the fraction will be increased by $\frac{9}{40}$. What is the fraction?

28. If 1 is added to the denominator and subtracted from the numerator of a certain fraction, the result will be the reciprocal of the fraction. If 1 is added to the numerator and subtracted from the denominator, the result will be $1\frac{3}{5}$ times the original fraction. Find the fraction.

---

29. A man bought some sheep in May for \$ 360. In June he bought 20 more than in May, paying \$ 2 less each, and the cost was \$ 320. How many did he buy in June?

30. Two trains start simultaneously from opposite ends of a double-track railroad 300 miles long. After they pass, one train takes 9 hours, the other 4 hours, to complete the journey. Find the speed of each train.

31. Eight persons contributed \$ 30 to a benevolent purpose. If one half of the amount was contributed by men and the other half by women, what did each man and each woman contribute, if each man gave \$ 2 more than each woman?

32. The difference of the contents of two cubical boxes is 342 cubic inches, and the difference of their edges is 6 inches. Find the dimensions of each box.

33. An encyclopædia consisting of a certain number of volumes cost \$ 100, but if the volumes had cost \$ 1 apiece less, I should have had as many more volumes for the money as the number of dollars paid for each volume. What was the cost of each volume?

34. A rectangular lot containing 270 square rods is surrounded by a road 1 rod wide. If the area of the road is 70 square rods, what are the dimensions of the lot?

35. The sum of the diagonal and the longer side of a rectangle is 3 times the length of the shorter side, and the difference in the lengths of the two sides is 4 yards. Find the area of the rectangle.

36. The sum of the squares of the two digits of a number is equal to the number increased by the product of the digits, and if 36 be added to the number, the digits will be reversed. What is the number?

37. The sum of the numerator and denominator of a fraction is 17. If the numerator be diminished by 2 and the denominator increased by 2, the product of the resulting fraction and the original fraction is $\frac{7}{24}$. Find the fraction.

38. Two rectangular fields each contain 1 acre. One of the fields is 4 rods shorter and 2 rods broader than the other. Required the dimensions of each field.

39. A man rowed 9 miles down stream and back in 8 hours. At another time he went over the same course at twice his former speed, making the trip in 3 hr. 12 min. Find his rate and that of the current.

40. In going 180 feet the fore wheels of a buggy make 5 revolutions more than the hind wheels, but if the circumference of each were increased by 3 feet, the fore wheels would make only 3 revolutions more than the hind wheels in the same distance. What is the circumference of each?

41. $200 amounted to $260 at a certain rate and time. If the time had been 1 year less and the rate 2% more, it would have amounted to $264. Find the time and rate.

42. A, B, and C began on the same day to solve a certain number of problems. A solved 10 a day, and finished them 4 days before B finished. B solved 2 more a day than C, and finished them 5 days before C. How many problems were solved, and how many days did it take each to finish the task?

43. A grocer buys $45 worth each of coffee and sugar, and receives 540 pounds more of the latter than of the former. He sells 50 pounds of sugar and 20 pounds of coffee at a profit of 20%, receiving $6. How many pounds of each did he buy?

44. A and B run a race. B, who runs slower than A by 1 mile in 2 hours, starts first by 2 minutes, and they get to the 4-mile stone together. Required rates of running.

45. Seventy-nine is the result of adding the sum of two numbers to their product, while 47 is the result of subtracting the sum from the product. Find the numbers.

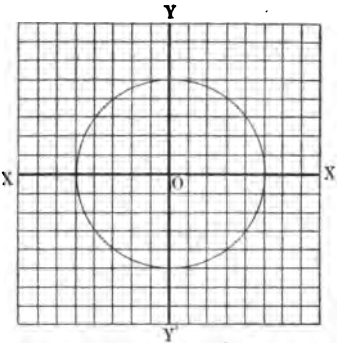
46. Two sums of money amounting to $9000 are loaned at such a rate of interest that the income from each is the same. But if the first had been at the same rate as the second, the income from it would have been $160; while if the second had been at the same rate as the first, the income from it would have been $250. Find the rate at which each sum is loaned.

GRAPHS OF QUADRATIC EQUATIONS

372. We have already seen that the graph of a simple equation containing two unknown numbers is always a straight line.

373. It has also been shown that just as two straight lines have in general but one point in common, so two linear equations have but one common pair of roots, the values of the roots being the same as the common point of the graphs.

We are now prepared to consider the graphs of quadratic equations.



374. Consider first the graph of the equation

$$x^2 + y^2 = 25.$$

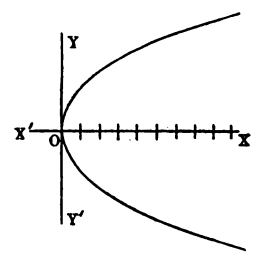
Solving for y, $y = \pm\sqrt{25 - x^2}$, from which it is seen that, for *real* values of y, $x^2 \not> 25$; that is, $x \not< -5$, or $x \not> +5$. Corresponding values of x and y are as follows:

$$x = 0, \quad \pm 1, \quad \pm 2, \ \pm 3, \ \pm 4, \ \pm 5;$$
$$y = \pm 5, \ \pm\sqrt{24}, \ \pm\sqrt{21}, \ \pm 4, \ \pm 3, \quad 0.$$

Plotting these pairs of points, using the approximate values ± 4.9 and ± 4.6 for $\pm\sqrt{24}$ and $\pm\sqrt{21}$ respectively, and drawing a continuous curve through the points, the graph is found to be a **Circle** of radius 5 and center at 0.

This curve is characteristic of graphs of equations of the type $x^2 + y^2 = r^2$.

375. Consider the graph of the equation $y^2 = 4x$.



Solving for y, $y = \pm 2\sqrt{x}$, from which it is seen that $x \not< 0$ for real values of y. For

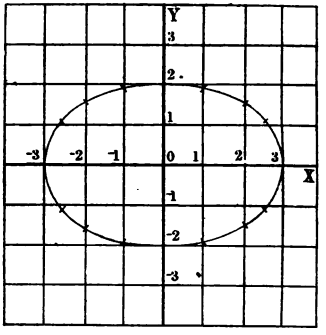
$x = 0, \ 1, \ 2, \ 3, \ 4, \ 5, 6, \cdots,$
$y = 0, \ \pm 2, \ \pm 2\sqrt{2}, \ \pm 2\sqrt{3}, \ \pm 4, \ \cdots,$
$\ \ = 0, \ \pm 2, \ \pm 2.8, \ \pm 3.4, \ \pm 4, \ \cdots.$

Plotting these pairs of points, and passing a continuous curve through them, we obtain the curve in the annexed figure. The graph is called a **Parabola**.

This curve is characteristic of graphs of equations of the type $y^2 = mx$.

376. Consider the graph of the equation

$$4x^2 + 9y^2 = 36.$$



Solving for y, $y = \pm \frac{2}{3}\sqrt{9 - x^2}$, from which it is seen that $x^2 \not> 9$; that is, $x \not< -3$ or $x \not> +3$, for real values of y.

For $x = 0, \quad \pm 1, \quad \pm 2, \quad \pm 3,$

$y = \pm 2, \ \pm \frac{4}{3}\sqrt{2}, \ \pm \frac{2}{3}\sqrt{5}, \ 0,$

$= \pm 2, \ \pm 1.8, \quad \pm 1.4, \quad 0.$

Plotting these pairs of points, and passing a continuous curve through them, we obtain the curve here given. The graph is called an **Ellipse**.

It is noticed that the extreme right and left hand portions of the curve are but roughly determined by the given points. To determine these portions we may assume any intermediate values of x between ± 2 and ± 3 and calculate the corresponding values of y, as the point $(\pm 2.5, \pm 1.1)$.

The curve last given is characteristic of graphs of equations of the type $ax^2 + by^2 = c$, where a, b, c are positive.

377. Consider the graph of the equation

$$4x^2 - 9y^2 = 36.$$

Solving for y,

$$y = \pm \tfrac{2}{3}\sqrt{x^2 - 9},$$

from which it is seen that for real values of y, $x^2 \not< 9$.

For

$x = \pm 3, \quad \pm 4, \quad \pm 5,$

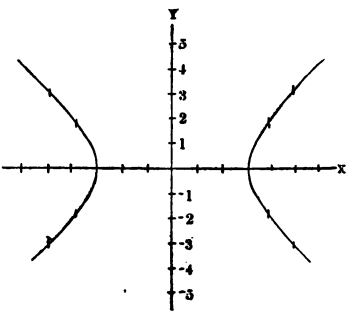
$\pm 6, \ \cdots,$

$y = 0, \quad \pm \frac{2}{3}\sqrt{7}, \ \pm \frac{2}{3} \times 4,$

$\pm \frac{2}{3}\sqrt{27}, \ \cdots,$

$= 0, \quad \pm 1.7, \quad \pm 2.6,$

$\pm 3.4, \ \cdots.$



Plotting these pairs of points, and passing a continuous curve through them, we obtain the two-branched curve here given. The graph is called an **Hyperbola.**

This curve is characteristic of the graphs of equations of the type $ax^2 - by^2 = c$, where a, b, c are positive.

Draw the graphs of the following equations:

1. $x^2 + y^2 = 4$.
2. $y^2 = 8x$.
3. $y = 2x^2 + 1$.
4. $4x - y^2 = 16$.
5. $y^2 = 4x + 2$.
6. $y^2 = 2x - 6$.
7. $3x^2 - 4y^2 = 12$.
8. $x^2 - y^2 = 25$.
9. $y = x^2 - 6x - 3$.
10. $25x^2 + 4y^2 = 100$.

In how many places can each of these curves be cut by a straight line?

Find, to the same set of axes, the graphs of each pair of the following equations:

11. $x^2 + y^2 = 25$.
    $xy = 12$.
12. $x^2 + y^2 = 25$.
    $x - 5y = 0$.
13. $x^2 + y^2 = 5$.
    $y + 2x = 3$.
14. $y = 3x - 12$.
    $x^2 - y^2 = 16$.
15. $4x^2 - 9y^2 = 36$.
    $2x + 3y = 6$.
16. $4x^2 + 25y^2 = 100$.
    $x^2 + y^2 = 9$.

378. Representation of the Roots.

In the equation $y = x^2 + 2x - 4$,

the corresponding values of x and y are conveniently computed as follows:

$$x = 3,\ 2,\ 1,\ 0,\ -1,\ -2,\ -3,\ -4,\ -5. \quad (1)$$

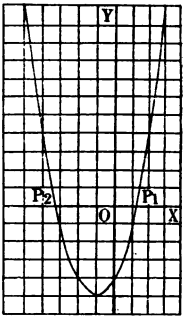
$$x^2 = 9,\ 4,\ 1,\ 0,\ 1,\ 4,\ 9,\ 16,\ 25.$$

$$x^2 + 2x = 15,\ 8,\ 3,\ 0,\ -1,\ 0,\ 3,\ 8,\ 15.$$

$$y = x^2 + 2x - 4 = 11,\ 4,\ -1,\ -4,\ -5,\ -4,\ -1,\ 4,\ 11. \quad (2)$$

Plotting the pairs of points in (1) and (2), and passing a curve through them, we obtain the curve given in the figure.

Now, to solve the quadratic equation $x^2 + 2x - 4 = 0$, consists in finding such values for x as will render the first member 0, and if we write $x^2 + 2x - 4 = y$, then it consists in finding the values of x which make $y = 0$. In the graph it consists in finding the abscissas of the points in which the graph meets the X-axis, since $y = 0$ for these points.



When $y = 0$, $x = -1 + \sqrt{5}$ and $-1 - \sqrt{5}$, or approximately 1.2 and -3.2, these values of x being the abscissas of the points in which the graph meets the X-axis.

Hence, it is seen that any quadratic equation of the form $x^2 + bx + c = 0$ can be solved graphically by writing $x^2 + bx + c = y$ and plotting the graph, and then measuring the abscissas of the points in which the graph meets the X-axis, attaching the proper signs to the values found.

379. When an equation has no real roots the graph does not reach the X-axis. For instance, the graph of the equation $y = x^2 - 6x + 10$, as shown in Fig. 1, does not reach the X-axis, the roots of the equation $x^2 - 6x + 10 = 0$ being *conjugate complex numbers*. It will be observed that the curve is symmetric with respect to a line $x = 3$.

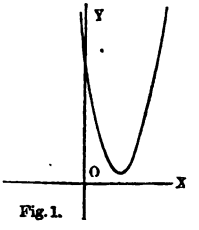


Fig. 1.

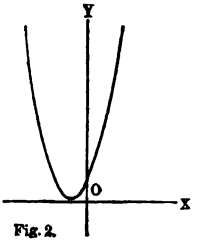


Fig. 2.

When an equation has a double root its graph touches the X-axis but does not cross it. For, plotting the graph of $y = x^2 + 2x + 1$ and solving $x^2 + 2x + 1 = 0$, we find that the latter has the roots -1 and -1 and that the graph does not cross the X-axis.

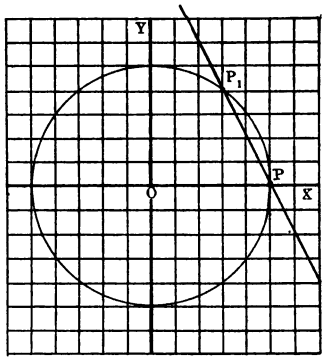
380. Simultaneous Quadratic Equations.

We have already seen that to solve two simultaneous linear equations containing two unknown numbers is equivalent to finding the coördinates of the points of intersection of the graphs of the two equations. We shall now see that the coördinates of the common points of two simultaneous quadratic equations containing two unknown numbers also represent the common roots of the equations, and that, in general, such equations have four roots, the graphs intersecting in four points.

381. For simplicity first consider the system

$$\begin{cases} x^2 + y^2 = 25, \\ 2x + y = 10, \end{cases}$$

in which one equation is *linear*. If we draw the graphs of the two equations on the same diagram, as in the annexed figure, we find that the first equation is represented by a circle, and the second by a straight line, and that the graphs intersect at the points P and P_1 whose coördinates are respectively (3, 4) and (5, 0).



Solving the system by the method given in Art. 364, we find

$$x = 3,\ 5,$$
$$y = 4,\ 0,$$

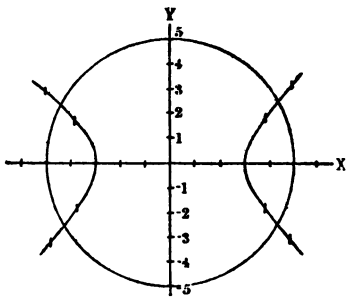
the roots already found by the graphic method.

It is to be noticed that in this case there are only *two common pairs of roots*. At how many points can a straight line cut the circumference of the circle?

382. Consider the system $\begin{cases} x^2 + y^2 = 25. & (1) \\ 4x^2 - 9y^2 = 36. & (2) \end{cases}$

The graph of (1) is shown in Art. 374, and that of (2) in Art. 377, the former being a circle and the latter an hyperbola.

Plotting both of the graphs on the same diagram, as in the annexed figure, we find that the curves intersect in the four points (4.5, 2.2), (4.5, −2.2), (−4.5, 2.2), (−4.5, −2.2). These are the four pairs of roots found by solving the system by eliminating x and y by addition and subtraction. How, then, do the points of intersection compare with the common roots of the two equations?



Draw the graphs of the following systems of simultaneous equations, and ascertain that, for real values, they intersect in the points given by the solution:

1. $x + y = 2.$
   $xy = -15.$

2. $x^2 - 3y = 10.$
   $9x^2 + 25y^2 = 225.$

3. $x^2 + y^2 = 9.$
   $4x^2 + 9y^2 = 36.$

4. $x^2 + y^2 = 16.$
   $x^2 + 4y^2 = 64.$

5. $y + 2x = 5.$
   $x^2 + y^2 = 5.$

6. $y^2 = 4x.$
   $x - y = -1.$

7. $4x^2 + 9y^2 = 36.$
   $x + 3y = 5.$

8. $xy = 12.$
   $x^2 + y^2 = 40.$

9. $3x^2 + 4y^2 = 24.$
   $x^2 - y^2 = 4.$

10. $2x^2 + 3y^2 = 1.$
    $2x - 3y = -3.$

11. $x^2 + 3y^2 = 28.$
    $2x^2 - y^2 = -7.$

12. Draw the graphs and explain the solution of

$$y = 2x - 3; \quad y = \frac{x^2}{4} + x - 2.$$

INEQUALITIES

383. When subtracting one number from another gives a *positive* remainder, the minuend is said to be *greater* than the subtrahend; it is said to be *less* than the subtrahend when the remainder is *negative*.

Thus, when $a - b$ is *positive*, a is greater than b; when $a - b$ is *negative*, a is less than b. Again, $-3 > -5$ because $-3 - (-5)$ is positive, but $-5 < -3$ because $-5 - (-3)$ is negative.

384. An **Inequality** is a statement that one number or expression is greater or less than another.

385. If two inequalities have the same sign, they are said to *subsist in the same sense;* if they have opposite signs, they are said to *subsist in an opposite sense.*

Thus, $a > b$ and $c > d$ subsist in the same sense, but $a > b$ and $c < d$ subsist in the opposite sense.

It is assumed in the present discussion, except as the minus sign indicates a negative number, that the letters denote positive and real finite numbers.

386. Since the statement that a is greater than b implies the statement that b is less than a, it follows that

The members of an inequality may be interchanged provided the sign of inequality be changed.

Thus, the relations $m > n$ and $8 > 5$ may be stated as follows: $n < m$ and $5 < 8$.

387. Changing the signs of all the terms of an inequality necessitates the changing of the sign of inequality.

Thus, $6 > 2$, but $-6 < -2$. Why?

388. If the same number or equal numbers are added to or subtracted from each member of an inequality, the resulting inequality will subsist in the same sense.

For, if $a > b$

then $a - b$ is positive.

$\therefore (a + c) - (b + c)$ is positive. (Why?)

But if $(a + c) - (b + c)$ is positive,

then $(a + c) > (b + c)$.

Similarly, $(a - c) > (b - c)$.

Thus, if $5 > 2$, $5 + 1 > 2 + 1$; if $-3 > -7$, $-3 - 2 > -7 - 2$.

389. If both members of an inequality are multiplied or divided by the same positive number or by equal positive numbers, the resulting inequality will subsist in the same sense, but it will subsist in the opposite sense if the multiplier or divisor is negative.

For, if $a > b$,

then $a - b$ is positive.

$\therefore m(a - b)$, or $ma - mb$ is positive.

But if $ma - mb$ is positive,

then $ma > mb$.

Again, if $a - b$ is positive,

then $-m(a - b)$ is negative,

that is, $-ma - (-mb)$ is negative.

But if $-ma - (-mb)$ is negative,

then $-ma < -mb$.

Thus, if $8 > 6$, then $8 \times 2 > 6 \times 2$, $8 \div 2 > 6 \div 2$, $8 \times -2 < 6 \times -2$.

Let the student put $\frac{1}{m}$ for m, and prove the principle for division.

390. If the corresponding members of any number of inequalities subsisting in the same sense are multiplied together, the resulting inequality will subsist in the same sense, provided all the members are positive.

If $a > b$, $c > d$, $e > f$, then $ace > bdf$, all the members being positive. Show that this is true when $a = 8$, $b = 7$, $c = 6$, $d = 5$, $e = 4$, $f = 3$.

391. It follows from Art. 390 that, when a and b are positive numbers and n is a positive integer, if

$$a > b, \text{ then } a^n > b^n.$$

Thus, $16 > 9$, $256 > 81$.

392. An inequality that holds for any values that may be given to the literal numbers involved is said to be *identical.*

Thus, $m^2 + n^2 > m^2$ is an *identical inequality*, and is analogous to an *identity.*

An inequality that holds only for certain values of the literal numbers involved is said to be *conditional.*

Thus, $x + 3 > 4$ is a *conditional inequality*, since it holds only for values of x greater than 1.

393. The sum of the squares of two unequal numbers is greater than twice the product of the numbers; that is, if $a \neq b$, then $a^2 + b^2 > 2ab$.

For $(a - b)^2 > 0$, since the square of any real number is positive, and therefore greater than zero.

$$\therefore a^2 - 2ab + b^2 > 0.$$

$$\therefore a^2 + b^2 > 2ab. \qquad \text{(Why ?)}$$

If $a = b$, how does $a^2 + b^2$ compare with $2ab$?

1. Solve the inequality $6x - 10 > 14$.

$$6x - 10 > 14.$$

Adding 10, $\quad 6x > 24.$

Dividing by 6, $\quad x > 4.$

That is, x cannot have any value as small as 4, but may have any value greater than 4.

Solve the following inequalities:

2. $3x-8>7$.

3. $2x+3<12$.

4. $5x<8x-6$.

5. $8x-3>3x+1$.

6. $(x+2)^2>x^2+3x+1$.

7. $7x-4>\frac{5x}{2}+\frac{6x}{4}$.

8. Find the numbers between which the value of x lies in the inequalities $4x+8<20$ and $3x<5x-1$.

We have given	$4x+8<20$.
By Art. 388,	$4x<12$.
By Art. 389,	$x<3$.
We have given also	$3x<5x-1$.
By Art. 388,	$-2x<-1$.
By Art. 387,	$2x>1$.
By Art. 389,	$x>\frac{1}{2}$.

Since $x<3$ and $x>\frac{1}{2}$, the value of x lies between $\frac{1}{2}$ and 3. The former is called the least, or *minimum*, value of x; the latter is called the greatest, or *maximum*, value.

Solve the following inequalities:

9. $4x-6>2$.
$3x-9<3$.

10. $4x-3>1$.
$5x+5<15$.

11. $6x>-1$.
$4x-25<0$.

12. $3x-9>6$.
$7x-40<9$.

13. What values of x satisfy the inequality $x^2+5x>24$?

By Art. 388,	$x^2+5x-24>0$.
Factoring,	$(x+8)(x-3)>0$.

If the product $(x+8)(x-3)$ is positive, both factors are positive or both are negative. Both are positive, *i.e.*, greater than 0, if $x>3$. Why? Both are negative if $x<-8$. Why?

Therefore, x can have any values *greater than* 3 and *less than* -8; that is, $x>3$ and $x<-8$.

Find the maximum and minimum values of x in the following:

14. $x^2 - x > 6$.
15. $x^2 + 4x > 21$.
16. $x^2 - 9x > -18$.
17. $(x-4)(5-x) > 0$.
18. $x^2 + 32 > 12x$.
19. $x^2 + 9x - 10 > 0$.
20. $x^2 + (b-a)x > ab$.
21. $x^2 + 50x > 5(3x + 40)$.

22. What values of x and y satisfy simultaneously the equation

$$2x - 3y = 2,$$

and the inequality $3x + 4y > 11.5$?

Multiplying the inequality by 2, $6x + 8y > 23$. (1)

Multiplying the equation by 3, $6x - 9y = 6$. (2)

Subtracting (2) from (1), $17y > 17$.

$\therefore\ y > 1$. (dividing by 17)

To eliminate y, multiply the inequality by 3.

$$\therefore\ 9x + 12y > 34.5, \tag{3}$$

and multiplying the equation by 4,

$$8x - 12y = 8. \tag{4}$$

Adding (3) and (4), $17x > 42.5$.

$\therefore\ x > 2.5$. (dividing by 17)

By assigning to x any value greater than 2.5 and solving the equation for y, we have y greater than 1. (Show this. Let $x = 4$, and find value of y.)

Find values of x and y that will satisfy the following systems:

23. $x + 2y = 10$.
$2x - y > 0$.

24. $2x - y = 7$.
$3x - 3y > 3$.

25. $2x = 2y + 10$.
$3x < 7y$.

26. $y - 4 = 3x$.
$2y + 3x > 25$.

27. Show that the sum of any positive number n and its reciprocal is greater than 2, except when $n = 1$.

If $n =$ any positive number,

then $\frac{1}{n} =$ its reciprocal.

Then $n + \frac{1}{n} = \frac{n^2+1}{n}$.

But $\frac{n^2+1}{n} > \frac{2n}{n}$, or 2. (Art. 393)

What is the result when $n = 1$?

If the literal numbers are unequal and positive, show that:

28. $\frac{a+3b}{a+2b} > \frac{a+5b}{a+4b}$.

29. $\frac{1}{2}(a+b) > \sqrt{ab}$.

30. $\frac{b}{a} < \frac{d}{c}$ if $\frac{a}{b} > \frac{c}{d}$.

31. $2a^3 > a+1$, if $a > 1$.

32. $ab + bc + ac < a^2 + b^2 + c^2$.

33. $a^3 + 1 > a^2 + a$, if a is a negative proper fraction.

34. $\frac{a}{b} + \frac{b}{a} < 2$, if a and b are of opposite qualities.

35. $\frac{a^2+b^2}{a+b} < \frac{a^3+b^3}{a^2+b^2}$, if a and b are unequal.

36. $a - b > (\sqrt{a} - \sqrt{b})^2$, if $a > b$.

37. Twice the number of pupils in a class diminished by 7 is greater than 29; and 3 times the number diminished by 5 is less than twice the number increased by 16. How many pupils in the class?

38. Five times a certain number diminished by 70 is less than 4 times the number diminished by 9; and 3 times the number increased by 2 exceeds twice the number by 61. What is the number?

39. A man does not have enough money to pay for a $20 watch, but if he borrows half as much as he has, he can pay for it and have more money left than he now lacks. How much money has he?

RATIO, PROPORTION, AND VARIATION

RATIO

394. The relation of one number to another of the same kind is **Ratio**, when it is expressed by the quotient of the first number divided by the second.

Thus, by the ratio of two algebraic numbers a and b is meant the quotient of a divided by b. Why is there no ratio between \$ 3 and 6 *hats*? In algebra the letters used always represent *abstract* numbers.

395. The symbol of ratio is (:).

Thus, the ratio of a to b is written $a : b$. But ratio may also be expressed in the fractional form, as $\frac{a}{b}$.

396. The first of the two numbers compared is called the **Antecedent**, the second the **Consequent**.

Thus, in the ratio $a : b$, a is the antecedent and b the consequent. It follows from the definition that the ratio of one number to another is found by dividing the antecedent by the consequent. Why is the ratio always an *abstract* number?

397. If the antecedent and consequent are interchanged, the resulting ratio is called the *inverse* of the given ratio.

Thus, the ratio $b : a$ is called the inverse of the ratio $a : b$.

398. If the ratio of two numbers is expressible as a rational number, the numbers are said to be **Commensurable**; otherwise they are said to be **Incommensurable**.

Thus, the diagonal and side of a square are incommensurable, their ratio being expressed by $\sqrt{2}$, an irrational number. Are the diameter and circumference of a circle commensurable?

399. PRINCIPLE. **Multiplying or dividing both antecedent and consequent by the same number does not change the ratio.**

For the ratio $a:b$ is identical with $\frac{a}{b}$, and by Art. 176,

$$\frac{a}{b}=\frac{am}{bm}.$$

$$\therefore\ a:b=am:bm.$$

Are all the properties of fractions true of ratios? Why?

1. The ratio of x^2 to 5 is 20. What is the value of x?

By definition, $\frac{x^2}{5}=20.$

$$\therefore\ x^2=100.$$

$$\therefore\ x=\pm 10.$$

2. If $a<b$, prove that the ratio of $a:b$ is increased by adding any positive number, m, to both antecedent and consequent.

It is to be proved that $\frac{a+m}{b+m}>\frac{a}{b}$ if $a<b$.

We have given $a<b$.

By Art. 389, $am<bm$.

Adding ab, $ab+am<ab+bm$,

or $a(b+m)<b(a+m)$.

Dividing by $b+m$, $a<\frac{b(a+m)}{b+m}$.

Dividing by b, $\frac{a}{b}<\frac{a+m}{b+m}$.

What is the effect of adding m to both antecedent and consequent when $a>b$? When $a=b$? If 2 is added to both numerator and denominator of $\frac{3}{8}$, what is the effect on the value? If 2 is added to both numerator and denominator of $\frac{5}{3}$, what is the effect on the value? What conclusion may be drawn from a comparison of these examples?

Find the ratio of:

3. 3 to 5.

4. 2.5 to .25.

5. $\frac{4}{7}$ to $\frac{2}{7}$.

6. $6b$ to $3b$.

7. $\frac{a}{b}$ to $\frac{c}{d}$.

8. $x-3$ to x^3-27.

Find the ratio of x to y if:

9. $3x-2y=0$.

10. $\dfrac{4x+5y}{3x-y}=2$.

11. $\dfrac{3x-2y}{4x-3y}=\dfrac{a}{b}$.

12. $15(2x^2-y^2)=7xy$.

Find the value of x from the ratios:

13. $16:x^2=25$.

14. $\dfrac{x}{5}=\dfrac{20}{x}$.

15. $\dfrac{x^2}{3}=27$.

16. $\dfrac{3}{x^2}=\dfrac{1}{72}$.

17. Which is greater, $3:4$, $5:7$, $11:14$?

18. Which is greater, $\frac{3}{4}$, $2\div 3$, or $1:2$?

19. Which is the greater ratio, $\dfrac{a+3b}{a+4b}$ or $\dfrac{a+4b}{a+5b}$?

20. Divide 56 into two parts whose ratio shall be $2:3$.

21. What is the effect produced on the ratio $4:5$ by multiplying the antecedent by 3? The consequent by 3? Both by 3?

22. If a is the antecedent and 2 the value of ratio, what is the consequent?

23. What number must be added to both antecedent and consequent of the ratio $\frac{5}{14}$ to make the ratio $\frac{3}{4}$?

24. What number must be subtracted from both antecedent and consequent of the ratio $3:5$ to produce the ratio $5:9$?

25. A certain ratio becomes $\frac{3}{4}$ when 4 is added to both antecedent and consequent, and $\frac{1}{2}$ when 2 is subtracted from each. Find the ratio.

26. If $a>b$, will the ratio $a:b$ be increased or diminished by adding any positive number, m, to both antecedent and consequent?

27. If $a>b$, will the ratio $a:b$ be increased or diminished by subtracting any positive number, m, not greater than b, from the antecedent and consequent?

PROPORTION

400. An expression of the equality of two ratios is called a **Proportion.**

Thus, $4 : 8 = 6 : 12$, $\frac{a}{b} = \frac{c}{d}$ are proportions. The sign : : is sometimes written between the equal ratios instead of the sign =.

401. In the proportion $a : b = c : d$, a and d are called the **Extremes**, b and c the **Means.**

The four numbers a, b, c, and d are *proportionals*, or *in proportion*, since the first, a, is to the second, b, as the third, c, is to the fourth, d.

402. A proportion that consists of three or more equal ratios is often called a **Continued Proportion.**

Thus, $2 : 4 = 3 : 6 = 5 : 10$ is a continued proportion.

403. The proportion $a : b = b : c$, in which the consequent of the first ratio is equal to the antecedent of the second, is a special form of a proportion. In this case b is said to be a *mean proportional* between a and c, while c is said to be a *third proportional* to a and b.

404. Since the two ratios of a proportion form an equation, each member of which may be expressed as a fraction, it follows that the general laws governing the transformation and solution of equations apply in dealing with proportions.

PRINCIPLES OF PROPORTION

405. In any proportion the product of the extremes equals the product of the means.

For, if $a : b = c : d$,

$$\frac{a}{b} = \frac{c}{d}.$$

Multiplying by bd, $ad = bc.$ (1)

It follows from the preceding that in the proportion

$$a : b = b : c,$$

$$b^2 = ac.$$

$$\therefore\ b = \pm\sqrt{ac}.$$

That is, the mean proportional between any two numbers equals the two square roots of their product.

Dividing each member of equation (1) by d,

$$a = \frac{bc}{d}.$$

Dividing each member by c,

$$b = \frac{ad}{c}.$$

Hence, it is seen that an extreme may be found by dividing the product of the means by the other extreme, and a mean may be found by dividing the product of the extremes by the other mean.

Have the student give numerical illustrations.

406. If the product of two numbers equals the product of two other numbers, either two may be made the extremes, and the other two the means of a proportion.

For, by Art. 405, $ad = bc.$ (1)

Dividing by bd, $\frac{a}{b} = \frac{c}{d}$, or $a : b = c : d.$ (2)

This is evidently the converse of the principle stated in Art. 405.

Again, dividing (1) by cd, $\frac{a}{c} = \frac{b}{d}$, or $a : c = b : d.$ (3)

Dividing (1) by ab, $\frac{d}{b} = \frac{c}{a}$, or $d : b = c : a.$ (4)

Dividing (1) by ac, $\frac{d}{c} = \frac{b}{a}$, or $d : c = b : a.$ (5)

May (2) be written $c : d = a : b$? Why? May the ratios be interchanged in (3), (4), and (5)?

407. Transformation of a Proportion.

The following propositions relating to transformations which are possible in dealing with proportions are here established:

I. If $a : b = c : d$, then $a : c = b : d$.

For, if $$\frac{a}{b} = \frac{c}{d},$$

$$ad = bc. \qquad \text{(Art. 405)}$$

$$\therefore \frac{a}{c} = \frac{b}{d}, \text{ or } a : c = b : d. \text{ (Art. 406, (3))}$$

This proportion is said to be derived from the given proportion by **Alternation.**

II. If $a : b = c : d$, then $b : a = d : c$.

How does this follow from (5) in Art. 406?.

The second proportion is said to be derived from the first by **Inversion.**

III. If $a : b = c : d$, then $a + b : b = c + d : d$.

For, from $$a : b = c : d,$$

$$\frac{a}{b} = \frac{c}{d}.$$

Adding 1 to both members,

$$\frac{a}{b} + 1 = \frac{c}{d} + 1,$$

or $$\frac{a+b}{b} = \frac{c+d}{d}.$$

$$\therefore a + b : b = c + d : d. \qquad \text{(Why?)}$$

This result is said to be derived from the given proportion by **Composition.**

IV. If $a : b = c : d$, then $a - b : b = c - d : d$.

The proof is left as an exercise for the student.

The latter proportion is said to be derived from the former by **Division.**

V. If $a : b = c : d$, then $a + b : a - b = c + d : c - d$.

For, $$\frac{a+b}{b} = \frac{c+d}{d}, \quad \text{by III,} \tag{1}$$

and $$\frac{a-b}{b} = \frac{c-d}{d}, \quad \text{by IV.} \tag{2}$$

Dividing the members of (1) by the corresponding members of (2),

$$\frac{a+b}{a-b} = \frac{c+d}{c-d}.$$

This result is said to be derived from the given proportion by **Composition and Division.**

Practical application can often be made of this principle in solving equations in the fractional form, as will be seen in the solution of Ex. 4, Art. 408.

408. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

For, if $$\frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots,$$

we may put each of these equal ratios equal to k.

Then, $$\frac{a}{b} = k, \ \frac{c}{d} = k, \ \frac{e}{f} = k, \ \cdots.$$

$$\therefore a = kb, \ c = kd, \ e = kf, \ \cdots. \tag{1}$$

Adding the equations in (1),

$$a + c + e + \cdots = k(b + d + f + \cdots).$$

$$\therefore \frac{a + c + e + \cdots}{b + d + f + \cdots} = k = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} = \cdots, \quad \text{(Why?)}$$

or $$a + c + e + \cdots : b + d + f + \cdots = a : b = c : d = e : f = \cdots.$$

1. Find a mean proportional between 12 and 75.

$$12 : x = x : 75. \quad \text{(Art. 405)}$$

$$\therefore x^2 = 900.$$

$$\therefore x = 30.$$

2. If $a : b = c : d$, prove that

$$\frac{ma^2 + nb^2}{ma^2 - nb^2} = \frac{mc^2 + nd^2}{mc^2 - nd^2}.$$

If $a : b = c : d$, then by alternation (Art. 407, I),

$$a : c = b : d, \text{ or } \frac{a}{c} = \frac{b}{d}.$$

Let $$\frac{a}{c} = \frac{b}{d} = x.$$

Then $$a = cx \text{ and } b = dx.$$

By substitution, $$\frac{ma^2 + nb^2}{ma^2 - nb^2} = \frac{mc^2x^2 + nd^2x^2}{mc^2x^2 - nd^2x^2} = \frac{mc^2 + nd^2}{mc^2 - nd^2}.$$

3. What number must be subtracted from 9, 14, 16, 28 so that the numbers shall form a proportion ?

Let x = the required number.

By the conditions, $$\frac{9 - x}{14 - x} = \frac{16 - x}{28 - x}.$$

$$\therefore 252 - 37x + x^2 = 224 - 30x + x^2.$$

$$\therefore x = 4.$$

4. Solve the equation $\dfrac{\sqrt{x+5} + \sqrt{x-16}}{\sqrt{x+5} - \sqrt{x-16}} = \dfrac{7}{3}$.

By composition and division, $$\frac{2\sqrt{x+5}}{2\sqrt{x-16}} = \frac{10}{4}.$$

$$\therefore \frac{x+5}{x-16} = \frac{25}{4}.$$

$$\therefore 4x + 20 = 25x - 400.$$

$$\therefore x = 20.$$

Find the value of x if:

5. $6 : x = 3 : 7$.
6. $x : 6 = 3 : 9$.
7. $3 : 5 = x : 55$.
8. $2 : 9 = 7 : x$.
9. $10 : x = 8 : 4$.
10. $\frac{3}{8} : 2 = x : 1\frac{1}{2}$.
11. $-7 : 9 = -14 : x$.
12. $1 : a = a : x^2$.
13. $a : b = 7a : x$.
14. $3 + x : 4 + x = 5 : 6$.
15. $4 : 12 = x - 4 : x + 4$.
16. $x+1 : x-1 = x+2 : x-2$.

Find a fourth proportional to:

17. 1, 5, and 9. 19. a, b, and c. 21. $\frac{2}{3}$, $\frac{3}{5}$, $\frac{5}{6}$.

18. 16, 15, and 8. 20. x^3, xy, $5x^2y^2$. 22. $a-1$, a, and 1.

Find a mean proportional between:

23. 3 and 12. 25. a^4 and a^2b^2. 27. a^3b and ab^3.

24. 5 and 20. 26. $4ax^2$ and $16a^3$. 28. $\frac{1}{8}$ and $\frac{3}{4}$.

If $a:b=c:d$, prove that

29. $ma:mb=nc:nd$.

30. $a^n:b^n=c^n:d^n$.

31. $\dfrac{a^2+ab}{b^2-ab}=\dfrac{c^2+cd}{d^2-cd}$.

32. $\dfrac{a^2+ab}{c^2+cd}=\dfrac{b^2-2ab}{d^2-2cd}$.

33. $\dfrac{ab+cd}{ab-cd}=\dfrac{b^2+d^2}{b^2-d^2}$.

34. $\dfrac{a+c}{b+d}=\dfrac{c}{d}$.

35. $3a:2b=3c:2d$.

36. $c:d=\dfrac{1}{b}:\dfrac{1}{a}$.

37. $\dfrac{ac}{bd}=\dfrac{(a+c)^2}{(b+d)^2}$.

38. $\dfrac{5a+3b}{5a-3b}=\dfrac{5c+3d}{5c-3d}$.

39. $\dfrac{a^2-b^2}{a^2-3ab}=\dfrac{c^2-d^2}{c^2-3cd}$.

40. $\dfrac{a+b}{c+d}=\dfrac{\sqrt{a^2+b^2}}{\sqrt{c^2+d^2}}$.

41. Find a third proportional to $(a-b)^2$ and a^2-b^2.

42. If $a:b=b:c$, prove that

(1) $a:c=a^2:b^2$; (2) $a+b:b+c=\sqrt{a}:\sqrt{c}$.

43. If $a:b=b:c=c:d$, prove that

(1) $a:d=a^3:b^3$; (2) $a:c=a^2+b^2+c^2:b^2+c^2+d^2$.

44. If $a:b=c:d$, and $e:f=g:h$, prove that

$$ae:bf=cg:dh.$$

45. If $\dfrac{a}{b}=\dfrac{x}{y}$, prove that $\dfrac{a^2+b^2}{x^2+y^2}=\dfrac{a^2-b^2}{x^2-y^2}$.

46. If $\dfrac{x}{p-q}=\dfrac{y}{q-r}=\dfrac{z}{r-p}$, prove that $x+y+z=0$.

47. If $x+2:7=y+3:9$, find the ratio of $x-5$ to $y-6$.

48. If $\dfrac{x}{y}=\dfrac{3}{4}$, find the value of $\dfrac{5x-3y}{7x+2y}$.

Solve the equations, using methods of proportion:

49. $\frac{3x-7}{3x+7}=\frac{7}{3}$.

50. $\frac{5x-3a}{5x+3a}=\frac{7a-5}{13a-5}$.

51. $\frac{x^2+x-2}{x-2}=\frac{4x^2+5x-6}{5x-6}$.

52. $\frac{x^2-x+6}{x^2+x-6}=\frac{x^2+2x-3}{x^2-2x+3}$.

53. $\frac{\sqrt{x+b}+\sqrt{x-b}}{\sqrt{x+b}-\sqrt{x-b}}=a$.

54. $\frac{\sqrt{x+1}+\sqrt{x-1}}{\sqrt{x+1}-\sqrt{x-1}}=\frac{4x-1}{2}$.

55. $\frac{3+\sqrt{2x+3}}{5-\sqrt{2x+3}}=\frac{4+\sqrt{x+1}}{4-\sqrt{x+1}}$.

56. $\frac{\sqrt{x+a}+\sqrt{x-a}}{\sqrt{x+a}-\sqrt{x-a}}=\frac{m+n}{m-n}$.

57. Two numbers are in the ratio 2 : 5, and if 6 is added to each, they are in the ratio 4 : 7. Find the numbers.

58. What number must be added to 2, 3, 4, 7 so that the results shall form a proportion?

59. What number must be subtracted from 4, 5, 6, 8 so that the remainders shall form a proportion?

60. The ages of A and B are as 3 : 4, and 30 years ago they were as 1 : 3. What are their ages?

61. If $3a$ sheep cost \$ 24, how much will $5b$ sheep cost, if $a : b = 2 : 1$?

62. The sum of the extremes of a proportion is 23, the sum of the means 13, and the sum of the squares of the four proportionals 530. Find the proportion.

63. Divide 6 into two parts, such that their product shall be to the sum of their squares as 2 to 5.

64. The length of a rectangular field is to its width as 5 to 4, but if 4 rods be added to the length and 5 rods to the width, they will be to each other as 6 to 5. What is the area of the field?

65. In a mile race between two bicyclists, A and B, their rates were as 5 : 4. B had a half-minute start, but was beaten by 88 yards. Find the rate of each.

VARIATION

409. A number which may have different values in the same expression or equation is called a **Variable.** A number whose value is fixed is called a **Constant.**

Thus, in the equation $x + y = 5$, x and y are variables, since the equation can be satisfied by an unlimited number of values of the unknown numbers. But if $x - 1 = 2$, the value of x is fixed ; that is, $x = 3$.

Again, the expression for the area of any circle is πr^2. Here r is a variable, since it assumes different values in different circles, but π is a constant number.

410. Function of a Variable.

When one number is regarded as depending upon another for its value, the former is said to be a *function* of the latter; that is, a variable x is said to be a function of a variable y if to every value that is assigned to y there corresponds a definite value of x.

Thus, with a given principal and rate, the interest depends upon the time ; the interest is in this case a *function* of the time. The area, πr^2, of any circle is a function of the radius.

411. Direct Variation.

When two variable numbers, x, y, depend upon each other in such a way that when y is changed x is changed in *the same ratio*, x is said to *vary directly* as y. This is the same as saying that x is *proportional* to y.

Thus, if a man walks at the uniform rate of 3 miles an hour, he will walk 6 miles in 2 hours, 9 miles in 3 hours, 12 miles in 4 hours, and so on, so that the distance he walks varies as the time. That is, if x represents the number of miles he walks and y the number of hours occupied, then the relation between the distance in miles and the time in hours is expressed by the equation $x = 3y$, or $\frac{x}{y} = 3$.

412. By the definition, the expression "x varies directly as y" means that if y is doubled, halved, or changed in any other ratio, x is doubled, halved, or changed in the *same ratio.* It

follows that in such case the ratio of x to y always remains unchanged; that is, is constant. Hence, if $\frac{x}{y} = m$ (a constant), then $x = my$.

The **Sign of Variation** ($\propto$) is read "varies as." Thus, $x \propto y$ is read "x varies as y."

413. Inverse Variation.

If x varies directly as the *reciprocal* of y, it is said to vary *inversely* as y; that is, $x = m \times \frac{1}{y}$.

Thus, if a given sum of money is distributed among the poor of a city, the share of each person varies inversely as the number of recipients.

If the area of a rectangle be constant, the altitude varies inversely as the number representing the base; for, if the base is changed in any ratio, the area is changed in the same ratio unless the altitude is changed; if the latter is changed in an inverse ratio, the area is not changed.

414. If x varies directly as the product of y and z, it is said to vary *jointly* as yz; that is, $x \propto yz$, or $x = myz$.

Thus, with men of equal efficiency the amount of work done varies jointly as the number of workmen and the number of days. Doubling both quadruples the amount of work done.

If the base and altitude of a triangle both vary, the area of the triangle varies jointly as the product of their numerical values.

415. If x varies *directly* as y and *inversely* as z, then $x \propto \frac{y}{z}$, or $x = m \times \frac{y}{z}$.

Thus, the number of hours required to perform a journey varies directly as the number of miles to be traveled, and inversely as the rate of traveling. If the distance is doubled, the time will be doubled unless the rate is increased. If both rate and distance are doubled, the time remains unchanged.

The time required for some cattle to consume a stack of hay varies directly as the weight of the hay and inversely as the number of cattle. The greater the weight, the *longer* the time; the larger the number of cattle, the *shorter* the time.

416. We are now prepared to establish the following propositions in variation:

I. If $x \propto y$ and $y \propto z$, then $x \propto z$.

For, if $x \propto y$, then $x = my$, (by Art. 412); (1)

and, if $y \propto z$, then $y = nz$. (2)

$\therefore x = mnz$, by substituting (2) in (1).

$\therefore x \propto z$, mn being constant.

II. If $x \propto yz$, then $y \propto \frac{x}{z}$.

For $x = myz$. (Art. 414)

$\therefore y = \frac{x}{mz}$, or $\frac{1}{m} \times \frac{x}{z}$. (Why ?)

$\therefore y \propto \frac{x}{z}$, $\frac{1}{m}$ being constant.

For example, if the area of a rectangle varies as the product of the numbers representing the base and altitude, then the base varies as the quotient of the number representing the area divided by the number representing the altitude.

III. If $x \propto z$, and $y \propto z$, then $(x \pm y) \propto z$.

For $x = mz$ and $y = nz$.

$\therefore x \pm y = (m \pm n)z$. (Why ?)

$\therefore (x \pm y) \propto z$, $(m \pm n)$ being constant.

IV. If $x \propto z$ and $y \propto v$, then $xy \propto vz$.

For $x = mz$ and $y = nv$.

$\therefore xy = mnvz$, by multiplication.

$\therefore xy \propto vz$.

V. *If $x \propto y$ when z is constant, and if $x \propto z$ when y is constant, then $x \propto yz$ when both y and z are variable.*

Let a change in y to y' and a simultaneous change in z to z' together cause a change in x to x''. Since the variation of x depends on the variations of y and z, we may consider that the variations in y and z take place separately, each in turn producing a corresponding variation in x.

First, let x change to x', a value intermediate between x and x'', when y changes to y', z remaining constant.

Then $$\frac{x}{x'} = \frac{y}{y'}. \qquad (1)$$

Next, let x change from the value x' to the value x'' when z changes to z', y remaining constant at the value y'.

Then $$\frac{x'}{x''} = \frac{z}{z'}. \qquad (2)$$

Multiplying (1) by (2), we get

$$\frac{xx'}{x'x''} = \frac{yz}{y'z'},$$

or $$\frac{x}{x''} = \frac{yz}{y'z'}.$$

This shows that when yz changes to $y'z'$, x changes to x''. Hence, $x \propto yz$.

For example, the length of a journey varies as the number of miles traveled an hour when the time of performing the journey is constant, and varies as the number of hours occupied in the journey when the rate of traveling is constant, but varies jointly as the rate and time when both of them vary.

Again, the area of a triangle varies as the base when the altitude is constant, and as the altitude when the base is constant, but as the product of their numerical values when both base and altitude vary.

1. If $x \propto y$, and if $x = 3$ when $y = 7$, find x when $y = 12$.

Since $x \propto y$, then $x = my$.

$\therefore 3 = m \times 7$, and $m = \frac{3}{7}$.

$\therefore x = my = \frac{3}{7}y$.

When $y = 12$. $x = \frac{3}{7} \times 12 = \frac{36}{7}$, or $5\frac{1}{7}$.

2. If $x \propto \frac{1}{y}$, and if $x = 4$ when $y = 13$, find x when $y = 20$.

Since $x \propto \frac{1}{y}$, $x = m \times \frac{1}{y}$.

$\therefore 4 = m \times \frac{1}{13}$, and $m = 52$.

$\therefore x = 52 \times \frac{1}{20} = \frac{52}{20}$, or $2\frac{2}{5}$.

3. The volumes of spheres vary as the cubes of their radii. If three spheres of metal whose radii are 3 cm., 4 cm., and 5 cm., respectively, are melted into one, find the radius of the resulting sphere.

Let v stand for the number of cubic centimeters in the volume, and r for the number of centimeters in the radius of any sphere.

Then $$v \propto r^3, \text{ or } v = mr^3.$$

The volume of the resulting sphere is

$$m(3^3 + 4^3 + 5^3) \text{ ccm.} = 216\,m \text{ ccm.}$$

If v' denotes this volume, and r' the required radius, then

$$v' = mr'^3 = 216\,m.$$

$$\therefore r'^3 = 216, \text{ whence } r' = 6.$$

$\therefore$ the required radius is 6 cm.

4. The law of Boyle states that the volume of a given mass of any gas at a constant temperature varies inversely as the pressure. A mass of air at atmospheric pressure occupies 44 cubic inches. What is the volume when the pressure is $27\frac{1}{2}$ pounds a square inch, taking atmospheric pressure as 15 pounds on a square inch?

Let v represent the number of cubic inches of volume, and p the number of pounds of pressure.

Then, since $$v \propto \frac{1}{p}, \; v = m \times \frac{1}{p}.$$

$$\therefore 44 = m \times \tfrac{1}{15}, \text{ and } m = 660.$$

$$\therefore v = 660 \times \frac{1}{27\frac{1}{2}} = 24.$$

$\therefore$ the required volume is 24 cubic inches.

5. If $x \propto \dfrac{y}{z}$ and if $x = 12$ when $y = 2$ and $z = 3$, find the values of x when $y = 2$ and $z = 9$.

6. If $x \propto \dfrac{1}{y}$ and if $x = 6$ when $y = 8$, find the value of x when $y = 12$.

7. If $x \propto \dfrac{1}{y}$ and $y \propto \dfrac{1}{z}$, prove that $x \propto z$.

8. If $x \propto z$ and $y \propto \frac{1}{z}$, prove that $x \propto \frac{1}{y}$.

9. If $x \propto z$ and $y \propto z$, show that $\sqrt{xy} \propto z$.

10. If $x \propto s$ and $y \propto t$, prove that $\frac{x}{y} \propto \frac{s}{t}$.

11. If $x^2 \propto y$ and $z^2 \propto y$, then $xz \propto y$.

12. If $x + y \propto x - y$, prove that $x^2 + y^2 \propto xy$.

13. If $x \propto y^2$ and if $x = 1$ when $y = 5$, find the value of x when $y = 15$.

14. If $5x + 7 \propto 6y - 23$ and if $x = 6$ when $y = -3$, find the value of x when $y = 10$.

15. If $x \propto z$ and if $z = p$ when $x = q$, find the value of x when $z = q$.

16. If x varies directly as y and inversely as z, and if $x = 2$ when $y = 3$ and $z = 4$, find y when $x = 12$ and $z = 6$.

17. If $x \propto yz$ and if $x = 24$ when $y = 2$ and $z = 3$, find the value of y when $x = 8$ and $z = 4$.

18. If $x \propto y$ and if $x = 12$ when $y = 3$, find the value of y when $x = 32$.

19. The area of a circle varies as the square of the number representing its diameter (or radius). If the area of a circle whose diameter is 10 feet is 78.54 square feet, find the area of a circle whose diameter is 30 feet.

20. Prove that the area of the circle whose radius is 5 inches is equal to the sum of the areas of two circles whose radii are 3 inches and 4 inches, respectively.

21. If the weight of a wire varies directly as its length, and 1 meter of it weighs .8 grams, how much will 2.35 meters weigh?

22. The volume of a sphere whose radius is 1 foot is 4.1888 cubic feet. What is the volume of a sphere whose radius is 5 feet?

23. A cowboy uses a rope 7 feet long to tie his pony, thus allowing it to graze over 154 square feet of grass. If the rope were 10.5 feet long, over what area could it graze?

24. The weight of a sphere of given material varies as its volume, and its volume varies as the cube of the number representing its diameter. If a sphere 5 inches in diameter weighs 27 pounds, what is the weight of a sphere $6\frac{2}{3}$ inches in diameter?

25. If the volume of a soap bubble is 5.888 when the radius is 1 decimeter, find the volume when its radius is 1.5 decimeters.

26. The illumination from a given source of light varies inversely as the square of the number representing the distance. How much farther from a lamp must a book, which is now 20 inches off, be removed in order that it may receive half as much light?

27. The square of the number representing the time of a planet's revolution about the sun varies as the cube of the number representing its distance from the sun. If the distances of the Earth and Mercury from the sun are reckoned at 91,000,000 miles and 35,000,000 miles, respectively, find the time of Mercury's revolution.

28. The weight of a body varies inversely as the square of the number representing its distance from the earth's center. If a ball of metal weighs 20 pounds on the surface of the earth, what would be its weight at a distance of 8000 miles from the surface, reckoning the radius of the earth as 4000 miles?

29. The number of times a pendulum oscillates in a given time varies inversely as the square root of the number representing its length. If the length of a pendulum that oscillates once in a second is 39.2 inches, what is the length of a pendulum that oscillates twice in a second?

30. The distance through which a body falls from a state of rest varies as the square of the number representing the time of fall, and the velocity varies directly as the number representing the time. If a body falls 400 feet in 5 seconds, how far does it fall in 10 seconds? If the velocity at the end of 2 seconds is 64, what is the velocity at the end of 5 seconds?

SERIES

417. 1. In the series of numbers, 1, 2, 3, 4, 5, 6, ···, the difference between any two numbers is 1; that is, any number is 1 greater than the preceding. In the series 1, 3, 5, 7, 9, ···, what must be added to any number to obtain the next?

2. How may any number, or *term*, after the first be obtained from the preceding term of the series 2, 5, 8, 11, ···?

3. By how much does each term exceed the preceding term in the series 1, 5, 9, 13, ···? What is the seventh term of this series?

4. In the series 2, 4, 8, 16, ···, by what must any term be multiplied to obtain the next term? What is the sixth term?

5. If a series begins with 1 and increases by the constant multiplier 3, what is the third term? Write six terms of the series.

6. If a series begins with 2 and decreases by the constant multiplier $\frac{1}{2}$, what is the third term? Write five terms of this series.

7. Write four terms of a series beginning with a and increasing by a constant number d. By a constant multiplier r.

418. A **Series** is a succession of numbers in which each, after the first, is obtained from the preceding by some fixed law.

The single members of the series are called its **Terms**. The first and last terms are the **Extremes**, those intervening, the **Means**.

Thus, a, $a + d$, $a + 2d$, $a + 3d$, is a series, of which a and $a + 3d$ are the extremes.

ARITHMETICAL PROGRESSION

419. An **Arithmetical Series** is more commonly called an **Arithmetical Progression.** In it each term after the first is obtained from the preceding by the addition of a constant number called the **Common Difference.**

Thus, 3, 5, 7, 9, ⋯; 12, 9, 6, 3, ⋯; and a, $a+d$, $a+2d$, $a+3d$, ⋯, are arithmetical progressions. In the first the common difference is 2, and the series is *ascending;* in the second the common difference is -3, and the series is *descending;* in the third the common difference is d. Is this series ascending or descending? Does this depend upon whether d is *positive* or *negative?*

420. The nth Term.

When a is the first term of an arithmetical progression and d the common difference, the series is

$$a,\ a+d,\ a+2d,\ a+3d,\ a+4d,\ a+5d,\ \cdots.$$

Does the first term a appear in each term of the series? In the *third* term, how many times d is added to a? In the 4th term? The 6th? In any term, is the coefficient of d one less than the number of the term? If there are n terms, what is the coefficient of d in the nth term?

Letting l represent the nth, or last, term, we evidently have from the preceding,

$$l = a + (n-1)d. \qquad \text{(I)}$$

1. Find the 13th term of the series 2, 5, 8, 11, ⋯.

Here $a = 2$, $d = 3$, $n = 13$.

$$l = a + (n-1)d = 2 + (13-1) \times 3 = 38.$$

2. Find the 16th term of the series 1, 3, 5, 7, ⋯.
3. Find the 21st term of the series 2, 6, 10, ⋯.
4. Find the 19th term of the series 0, 6, 12, ⋯.
5. Find the 9th term of 3, -1, -5, -9, ⋯.
6. Find the 17th term of a, $6a$, $11a$, ⋯.
7. Find the 29th term of 1, $1\frac{1}{2}$, 2, $2\frac{1}{2}$, ⋯.
8. Find the 11th term of $a-b$, $a+b$, $a+3b$, ⋯.

421. If we know the *first* term and the common difference of a series, we may obtain all the terms by repeated additions of the common difference; and if the *last* term and common difference are known, we may obtain all the terms by repeated subtractions of the common difference.

Thus, if the first term is 2 and the common difference 3, we have 2, 5, 8, 11, 14, ···. And if the last term is 14, we have 14, 11, 8, 5, 2, ···. If the last term is l and the common difference d, we have l, $l-d$, $l-2d$, $l-3d$, $\cdots a$.

422. Sum of the Series.

If a represents the first term, l the last term, d the common difference, n the number of terms, and s the sum of n terms, it is evident that we may express *the sum of n terms* in two equivalent ways, as follows:

$$s = a + (a+d) + (a+2d) + (a+3d) + \cdots + l.$$

$$s = l + (l-d) + (l-2d) + (l-3d) + \cdots + a.$$

$$\therefore 2s = (a+l) + (a+l) + (a+l) + (a+l) + \cdots + (a+l)$$

$$= n(a+l).$$

$$\therefore s = \frac{n(a+l)}{2}. \qquad \text{(II)}$$

1. Find the sum of 16 terms of the series 1, 6, 11, ···.

From I, $l = a + (n-1)d = 1 + (16-1) \times 5 = 76.$

$$\therefore s = \frac{n}{2}(a+l) = \frac{16}{2}(1+76) = 616.$$

2. Find the sum of 10 terms of the series 2, 5, 8, ···.
3. Find the sum of 18 terms of the series 0, 4, 8, ···.
4. Find the sum of 22 terms of the series 1, $1\frac{1}{4}$, $1\frac{1}{2}$, ···.
5. Find the sum of 8 terms of the series -3, 1, 5, ···.
6. Find the sum of 10 terms of the series $3a$, a, $-a$, ···
7. Sum the series 5, 1, -3, ··· to 10 terms.
8. Sum the series $\frac{3}{8}$, $1\frac{1}{2}$, $2\frac{5}{8}$, ··· to 6 terms.
9. Sum the series n, $n+a$, $n+2a$, ··· to 8 terms.
10. Sum the series $m-n$, $m+n$, $m+3n$, ··· to 7 terms.

423. The two formulas,

$$l = a + (n-1)d, \tag{I}$$

$$s = \frac{n(a+l)}{2}, \tag{II}$$

are of *fundamental importance.* They involve five numbers, a, l, d, n, s, and it is evident that *any two* of these may be found by substituting in the formulas the known values of the *other three* and solving the resulting equations.

1. If the 5th term is -11 and the 9th term -27, find the first term and the common difference.

Here $n = 9$ and $n = 5$; hence,

From I, $a + 8d = -27$;

and $a + 4d = -11$.

Subtracting, $4d = -16$.

$\therefore d = -4$,

and $a = 5$.

2. The sum of n terms of the arithmetical progression 3, 9, 15, $\cdots$ is 363. Find n.

From I, $l = a + (n-1)d = 3 + (n-1) \times 6$.

$\therefore l = 6n - 3$.

Since $a = 3$, we have from II,

$$363 = \frac{n}{2}(a + l) = \frac{n}{2}(6n) = 3n^2.$$

$\therefore n^2 = 121$, and $n = 11$.

Let the student find the last term, l.

3. If the first term is 2, last term 34, and the sum of the series is 162 what is the common difference? The number of terms?

From I, $34 = 2 + (n-1)d$. (1)

From II, $162 = \frac{n}{2}(2 + 34)$. (2)

Solving (2) for n, we have $n = 9$.

Substituting 9 for n in (1), we have $d = 4$.

4. If $l = 59$, $d = 3$, and $s = 610$, find n and a.

From I, $$59 = a + (n - 1) \times 3. \quad (1)$$

From II, $$610 = \frac{n}{2}(a + 59). \quad (2)$$

Subtracting twice (2) from n times (1), we have

$$3n^2 - 121n = -1220;$$

whence $$n = 20.$$

Substituting 20 for n in (1), we have $a = 2$.

424. The table on the opposite page contains the formulas for all possible cases in arithmetical series. As an exercise the student should derive these results from formulas I and II, Art. 423.

425. The following problems may be solved by substituting in the formulas on p. 373, but it is better to substitute the given values in the two fundamental formulas, and then solve the resulting equations until this method has been made familiar to the student.

1. If the sum is 42, how many terms are there in the series $-3, 0, 3, \cdots$?

2. Find the 1st term and the common difference, if the 2d term is 9 and the 7th term 44.

3. If the first term is -11, last term 17, and the sum of the series 24, find the common difference and the number of terms.

4. Find the sum of the series $-12, -7\frac{1}{2}, -3, \cdots$ to 11 terms.

5. Find the first and last terms of the series $\cdots 6, 9, 12, \cdots$, if $n = 15$, and $s = 270$.

6. Find the sum of 12 terms of the series $-8, -2, 4, \cdots$.

7. If $s = 22\frac{1}{2}$, $d = 2\frac{1}{2}$, and $a = -2\frac{1}{2}$, find n.

8. Find the 20th term and the sum to 20 terms of the series $-3, -1, 1, \cdots$.

No.	Given	Required	Formulas
1	$a\ d\ n$	l	$l = a + (n-1)d.$
2	$a\ n\ s$		$l = \frac{2s}{n} - a.$
3	$d\ n\ s$		$l = \frac{s}{n} + \frac{(n-1)d}{2}.$
4	$a\ d\ s$		$l = -\frac{1}{2}d \pm \sqrt{2ds + (a - \frac{1}{2}d)^2}.$
5	$a\ d\ n$	s	$s = \frac{1}{2}n[2a + (n-1)d].$
6	$a\ n\ l$		$s = \frac{n}{2}(l + a).$
7	$a\ d\ l$		$s = \frac{l+a}{2} + \frac{l^2 - a^2}{2d}.$
8	$d\ n\ l$		$s = \frac{1}{2}n[2l - (n-1)d].$
9	$d\ n\ l$	a	$a = l - (n-1)d.$
10	$d\ n\ s$		$a = \frac{s}{n} - \frac{(n-1)d}{2}.$
11	$n\ l\ s$		$a = \frac{2s}{n} - l.$
12	$d\ l\ s$		$a = \frac{1}{2}d \pm \sqrt{(l + \frac{1}{2}d)^2 - 2ds}.$
13	$a\ n\ l$	d	$d = \frac{l-a}{n-1}.$
14	$a\ n\ s$		$d = \frac{2(s - an)}{n(n-1)}.$
15	$n\ l\ s$		$d = \frac{2(ln - s)}{n(n-1)}.$
16	$a\ l\ s$		$d = \frac{l^2 - a^2}{2s - l - a}.$
17	$a\ d\ l$	n	$n = \frac{l-a}{d} + 1.$
18	$a\ l\ s$		$n = \frac{2s}{l+a}.$
19	$a\ d\ s$		$n = \frac{d - 2a \pm \sqrt{(2a-d)^2 + 8ds}}{2d}.$
20	$d\ l\ s$		$n = \frac{2l + d \pm \sqrt{(2l+d)^2 - 8ds}}{2d}.$

9. How many terms of the series 80, 68, 56, ··· must be taken to make 248?

10. The extremes of an arithmetical progression are m and $-n$, the number of terms r. Find the sum.

426. A number that intervenes between two other numbers and forms with them an arithmetical progression is called their **Arithmetical Mean.**

Thus, 5 is an arithmetical mean between 2 and 8.

If A is the arithmetical mean between m and n, then m, A, n is an arithmetical progression.

$$\therefore A - m = n - A.$$

$$\therefore 2A = m + n, \text{ and } A = \frac{m+n}{2}.$$

427. All the terms between the extremes of a series are *arithmetical means.* Any number of arithmetical means may be inserted between two given numbers.

1. Insert four arithmetical means between 3 and 23.

The whole number of terms is 6. Why?

$$\therefore 23 = 3 + (6-1)d = 3 + 5d.$$

$$\therefore d = 4.$$

Hence the series is 3, 7, 11, 15, 19, 23.

Insert one arithmetical mean between:

2. 21 and 29.
3. a and $a + 4b$.
4. $x+1$ and $x-1$.
5. $x^2 - a$ and $a - x^2$.
6. Insert 5 arithmetical means between 7 and 25.
7. Insert 8 arithmetical means between 16 and -11.
8. Insert 2 arithmetical means between $m - n$ and $m + n$.
9. What is the arithmetical mean between x^2 and 1?
10. The first term of an arithmetical series is 2, and the sum of 5 terms is 40. What term is 11?
11. The 1st term of an arithmetical progression is -4, and the 4th term is 5. Find the sum of 9 terms.

12. How many strokes does a common clock strike in 24 hours?

13. The sum of 3 numbers in arithmetical progression is 15, and the sum of their squares is 83. What are the numbers?

Let the numbers be $x - y$, x, $x + y$.

Then $$(x - y) + x + (x + y) = 15, \quad (1)$$

and $$(x - y)^2 + x^2 + (x + y)^2 = 83. \quad (2)$$

From (1), $$x = 5. \quad (3)$$

From (2), $$3x^2 + 2y^2 = 83.$$

Substituting 5 for x, $$2y^2 = 8.$$

$$\therefore y = \pm 2.$$

Why is it convenient in this case to let $x - y$, x, $x + y$ represent the numbers?

14. The sum of 3 numbers in arithmetical progression is 15, and their product is 105. Find the numbers.

15. The sum of 3 numbers in arithmetical progression is 36, and the sum of their squares is 440. What are the numbers?

16. Find the sum of the odd numbers between 50 and 100.

17. In an arithmetical progression of 11 terms, the sum of the 3d and 8th terms is 20, and the sum of the 5th and 10th terms is 28. Write the series.

18. Find the sum of all the even numbers between 49 and 99.

19. The extremes of an arithmetical progression are 0 and 20, and the sum of the terms is 60. Write the series.

20. The sum of 4 numbers in arithmetical progression is 14, and the sum of their squares is 54. What is the series?

21. Find the sum of 20 consecutive odd numbers of which the greatest is 77.

22. If I save \$1 the first week, \$3 the second, \$5 the third, and so on, how much will I save in a year?

23. Insert arithmetical means between 5 and 41 so that the 4th of these means shall be to the next to the last, diminished by 1, as 1 to 2.

24. From the two fundamental formulas find n when d, l, and s are given.

25. If a body moves 4.9 meters in the first second, and in any succeeding second 9.8 meters more than in the preceding second, find the distance through which it moves in the sixth second.

26. A young man places \$160 at simple interest at 6% annually on the first day of each January for 5 years. What amount of principal and interest will be due him at the end of the 5 years?

27. A certain number consists of three digits which are in arithmetical progression. The hundreds' digit exceeds the sum of the tens' and the ones' digits by 1; and if 594 is subtracted from the number, the digits will occur in reverse order. Required the number.

28. A starts from a certain point, P, and travels 1 mile the first hour, 2 miles the second hour, 3 miles the third hour, and so on. Three hours later B starts in pursuit, traveling uniformly 10 miles an hour. After how many hours will B overtake A? How many hours later will A overtake B?

GEOMETRICAL PROGRESSION

428. A **Geometrical Series** is more commonly called a **Geometrical Progression.** In it each term after the first is obtained by multiplying the preceding term by a constant number called the **Ratio.**

Thus, 3, 6, 12, 24, and a, a^2, a^3, a^4, are geometrical progressions; in the first the ratio is 2, in the second it is a.

429. The *n*th Term.

If a is the first term, r the ratio, and n the number of terms, the series is $a, ar, ar^2, ar^3, ar^4, ar^5, \cdots ar^{n-1}$.

If the absolute value of r is greater than 1, the progression is an *ascending* one; otherwise it is a *descending* one.

Thus, 2, 3, $4\frac{1}{2}$, $6\frac{3}{4}$, ···, where $r = \frac{3}{2}$, and 3, -6, 12, -24, ···, where $r = -2$, are increasing series; while 2, $\frac{4}{3}$, $\frac{8}{9}$, $\frac{16}{27}$, ···, where $r = \frac{2}{3}$, and 3, $-\frac{3}{2}$, $\frac{3}{4}$, $-\frac{3}{8}$, ···, where $r = -\frac{1}{2}$, are decreasing series.

It will be observed that the exponent of r is always 1 less than the number of the term; in the 3d term the exponent is 2, in the 5th term it is 4, in the nth term it is $n - 1$.

It is therefore evident that if the nth, or last, term is represented by l, we have

$$l = ar^{n-1}. \qquad \text{(I)}$$

1. Find the 7th term of the series 2, 4, 8, ···.

Here $a = 2$, $r = 2$, and $n = 7$.

$$\therefore\ l = ar^{n-1} = 2 \times 2^6 = 128.$$

When $r = a$, $l = a^n$.

2. Find the 8th term of the series 1, 3, 9, ···.
3. Find the 9th term of the series 4, 8, 16, ···.
4. Find the 10th term of the series $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, ···.
5. Find the 12th term of the series $\frac{1}{3}$, $\frac{2}{3}$, $1\frac{1}{3}$, ···.
6. Find the 7th term of the series 5, 25, 125, ···.
7. Find the 6th term of the series 1, $\frac{3}{2}$, $\frac{9}{4}$, ···.
8. Find the 8th term of the series 2, -4, 8, ···.
9. Find the 6th term of the series $2a$, $-6ma^2$, $+18m^2a^3$, ···.

430. Sum of the Series.

If s represents the sum of n terms of a geometrical series, we have

$$s = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1}. \qquad (1)$$

$$\therefore\ rs = ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-1} + ar^n. \qquad (2)$$

$\therefore\ s(r-1) = ar^n - a$, subtracting (1) from (2).

$$\therefore\ s = \frac{ar^n - a}{r-1} = \frac{a(r^n-1)}{r-1}. \qquad \text{(II)}$$

Multiplying formula I by r, we have

$$lr = ar^n.$$

Substituting this value of ar^n in II, we have

$$s = \frac{lr - a}{r - 1}. \qquad \text{(III)}$$

When $r < 1$, formula II may be written

$$s = \frac{a(1 - r^n)}{1 - r}, \text{ and formula III, } s = \frac{a - lr}{1 - r}. \quad \text{(Why?)}$$

1. Find the sum of 7 terms of the series 2, 4, 8, ···.

Here $a = 2$, $r = 2$, and $n = 7$.

$$\therefore s = \frac{ar^n - a}{r - 1} = \frac{2 \times 2^7 - 2}{2 - 1} = 254.$$

2. Find the sum of 6 terms of the series 1, 3, 9, ···.
3. Find the sum of 9 terms of the series 4, 8, 16, ···.
4. Find the sum of 10 terms of the series $\frac{1}{8}$, $\frac{1}{4}$, $\frac{1}{2}$, ···.
5. Find the sum of 10 terms of the series 28, -14, 7, ···.
6. Sum the series $-\frac{1}{8}$, $\frac{1}{4}$, $-\frac{1}{2}$, ···, to 8 terms.
7. Sum the series 1, $2a$, $4a^2$, ··· to 6 terms.
8. Sum the series 1, r, r^2, ··· to 5 terms.
9. Sum the series 1, r^2, r^4, ··· to 6 terms.
10. Sum the series 1, a^2, a^4, ··· to n terms.
11. Sum the series 4, 12, 36, ··· to n terms.
12. Sum the series 2, 6, 18, ··· 486.
13. Sum the series 2, -4, 8, ··· -256.
14. If $a = 3$, $l = 192$, $r = 2$, find s.

431. Since the two *fundamental formulas* I (Art. 429) and II (Art. 430) may be treated as simultaneous equations in two unknowns in finding *any two* of the numbers a, r, l, n, s, when the *other three* are given, it is evident that various other formulas can be deduced from the given ones. However, the formulas for n, in general, require logarithms and may be taken up hereafter in connection with the study of Art. 483. In special cases the value of n can be found as in Exs. 2 and 3 below.

1. Write the series whose 4th term is 108 and 7th term 2916.

If a = the 1st term, and r = the ratio,

ar^3 = the 4th term, and ar^6 = the 7th term.

$$\therefore \frac{ar^6}{ar^3} = r^3 = \frac{2916}{108} = 27.$$

$$\therefore r = 3.$$

Since $ar^3 = 108$, $a = 108 \div 27 = 4$.

Hence the series is 4, 12, 36, ...

2. If $a = 1$, $r = 4$, and $s = 341$, find l and n.

From III, $$s = \frac{lr - a}{r - 1} = \frac{4l - 1}{3}.$$

$$\therefore 341 = \frac{4l - 1}{3}, \text{ whence } l = 256.$$

From II, $$s = \frac{a(r^n - 1)}{r - 1}, \text{ or } 341 = \frac{4^n - 1}{3},$$

whence $$1024 = 4^n.$$

Dividing by 4^4, $$4 = 4^{n-4}.$$

But $4 = 4^1$. $\therefore n - 4 = 1$, and $n = 5$.

3. The first term of a geometrical series is 3, the last term 96, and the sum of the series 189. Find the ratio and the number of terms.

From III, $$s = \frac{lr - a}{r - 1}, \text{ or } 189 = \frac{96r - 3}{r - 1},$$

whence $$r = 2.$$

From I, $$l = ar^{n-1}, \text{ or } 96 = 3 \times 2^{n-1}.$$

$$\therefore 2^{n-1} = 32, \text{ and } n = 6.$$

432. The table on page 380 contains the various formulas of geometrical series, and these should be worked out by the student from the two fundamental formulas, except those in which n is required. The latter are inserted in the table for reference, and may be taken up in connection with the next chapter.

No.	Given	Required	Formulas
1	$a\ r\ n$	l	$l = ar^{n-1}.$
2	$a\ r\ s$		$l = \dfrac{a + (r-1)s}{r}.$
3	$r\ n\ s$		$l = \dfrac{(r-1)sr^{n-1}}{r^n - 1}.$
4	$a\ n\ s$		$l(s-l)^{n-1} - a(s-a)^{n-1} = 0.$
5	$a\ r\ n$	s	$s = \dfrac{a(r^n - 1)}{r-1}.$
6	$a\ r\ l$		$s = \dfrac{rl - a}{r-1}.$
7	$r\ n\ l$		$s = \dfrac{lr^n - l}{r^n - r^{n-1}}.$
8	$a\ n\ l$		$s = \dfrac{\sqrt[n-1]{l^n} - \sqrt[n-1]{a^n}}{\sqrt[n-1]{l} - \sqrt[n-1]{a}}.$
9	$r\ n\ l$	a	$a = \dfrac{l}{r^{n-1}}.$
10	$r\ l\ s$		$a = rl - (r-1)s.$
11	$r\ n\ s$		$a = \dfrac{(r-1)s}{r^n - 1}.$
12	$n\ l\ s$		$a(s-a)^{n-1} - l(s-l)^{n-1} = 0.$
13	$a\ l\ s$	r	$r = \dfrac{s-a}{s-l}.$
14	$a\ n\ l$		$r = \sqrt[n-1]{\dfrac{l}{a}}.$
15	$a\ n\ s$		$r^n - \dfrac{s}{a}r + \dfrac{s-a}{a} = 0.$
16	$n\ l\ s$		$r^n - \dfrac{s}{s-l}r^{n-1} + \dfrac{l}{s-l} = 0.$
17	$a\ r\ l$	n	$n = \dfrac{\log l - \log a}{\log r + 1}.$
18	$a\ r\ s$		$n = \dfrac{\log[a + (r-1)s] - \log a}{\log r}.$
19	$a\ l\ s$		$n = \dfrac{\log l - \log a}{\log(s-a) - \log(s-l) + 1}.$
20	$r\ l\ s$		$n = \dfrac{\log l - \log[lr - (r-1)s]}{\log r} + 1.$

433. The following problems may be solved by substituting in the above formulas, or by using the two fundamental formulas. The latter method is to be preferred until the student has become acquainted with the process.

1. Find the 7th term of 1, 4, 16, ···.
2. If $a = \frac{1}{2}$ and $r = -2$, write 10 terms of the series.
3. If $a = 1$ and $r = -\frac{2}{3}$, write 7 terms of the series.
4. If $l = 320$, $r = 2$, and $n = 7$, find a and s.
5. Find the ratio when the 1st term is 5, the 3d term 180.
6. Find the sum of 7 terms of .2, 1, 5, ···.
7. Sum $4a - a + \frac{1}{4}a \cdots$ to 8 terms.
8. Find the 10th term and the sum to 10 terms of the series 4, -2, 1, ···.
9. Find the 3d term of the series of which x, y are the first two terms.

434. A number intervening between two other numbers, and forming with them a geometrical progression, is called their **Geometrical Mean.**

Thus, in the series 1, 4, 16, the geometrical mean between 1 and 16 is 4; it is -4 in the series 1, -4, 16.

If G is the geometrical mean between a and b, then a, G, b is a geometrical progression. By the definition of geometrical progression we have

$$\frac{G}{a} = r = \frac{b}{G}, \text{ whence } G^2 = ab.$$

$$\therefore G = \pm\sqrt{ab}.$$

1. Find the geometrical mean between 1 and $\frac{9}{16}$.

Here $G = \pm\sqrt{1 \times \frac{9}{16}} = \pm\frac{3}{4}$.

Find the geometrical mean between

2. 1 and $\frac{9}{4}$.
3. 2 and $\frac{2}{9}$.
4. $\frac{2}{4}$ and $\frac{8}{9}$.
5. 3 and $\frac{12}{49}$.
6. $\frac{5}{3}$ and $\frac{5}{27}$.
7. $\frac{a}{6}$ and $\frac{a^3}{24}$.

435. Several numbers intervening between two other numbers, and forming with them a geometrical progression, are called *geometrical means.*

Thus, in the series 1, 3, 9, 27, the geometrical means are 3 and 9.

1. Insert 3 geometrical means between 2 and 512.

There are 3 means and 2 extremes, making 5 terms in the series. The 1st term is 2, the 5th term 512.

From I, $l = ar^{n-1}$, or $512 = 2r^4$.

$\therefore r^4 = 256$, $r^2 = 16$, $r = \pm 4$.

$\therefore$ the series is 2, 8, 32, 128, 512,

or 2, -8, 32, -128, 512.

2. Insert 3 geometrical means between 2 and 162.

3. Insert 2 geometrical means between 5 and 135.

4. Insert 3 geometrical means between 2 and 1250.

5. Insert 5 geometrical means between $\frac{1}{2}$ and 2048.

6. Insert 4 geometrical means between $\frac{1}{4}$ and -8.

7. Insert 3 geometrical means between m^4 and n^4.

8. If a boy earns 5 cents on Monday, 10 cents on Tuesday, 20 cents on Wednesday, and so on for the week, how much does he earn on Saturday?

9. A foolish man agreed to buy 10 sheep, paying 10 cents for the first, 20 cents for the second, 40 cents for the third, and so on. What did the tenth sheep cost him?

10. What is the amount at compound interest of \$100 for 5 years at 6 per cent? (Here $a = \$100$, $r = 1.06$, and $n = 6$.)

11. By saving 1 cent the first day, 2 cents the second day, 4 cents the third day, and so on, doubling the amount every day, how much would be saved in the month of September in any year?

12. The population of a certain country in 1900 was 50 millions. If it doubles itself every 25 years, what will be the population in the year 2000?

13. In a geometrical series of 3 terms the sum of the extremes is 15, and the third term minus the second equals 6. Required the series.

Represent the series by x, xy, xy^2.

Then $$xy^2 + x = 15,$$

whence $$x = \frac{15}{y^2+1}. \qquad (1)$$

Also, $$xy^2 - xy = 6,$$

whence $$x = \frac{6}{y^2-y}. \qquad (2)$$

Equating (1) and (2), $$\frac{15}{y^2+1} = \frac{6}{y^2-y},$$

whence $$y = 2.$$

$$\therefore x = 3,\ xy = 6,\ xy^2 = 12.$$

14. In an increasing geometrical series of 4 terms, the difference of the extremes is 52, and the difference of the means is 12. Required the series.

15. Find 3 numbers in geometrical progression such that their sum is 62, and the sum of their squares is 2604.

16. The sum of 3 numbers in geometrical progression is 21, and the sum of their reciprocals is $\frac{7}{12}$. Find the numbers.

17. The sum of 4 numbers in geometrical progression is 15; the sum of the first and last exceeds the sum of the other two by 3. What are the numbers?

18. There are 3 numbers in geometrical progression whose sum is 62; and the sum of the first and second is to the sum of the second and third as 1 is to 5. Required the numbers.

19. Three numbers whose sum is 38 are in geometrical progression. If 1, 2, and 1 are added to them in order, the results will be in arithmetical progression. Required the numbers.

20. A man had \$3000 on Jan. 1, 1894, a year later had \$4000, and Jan. 1, 1896, had \$5333$\frac{1}{3}$. If he kept up this rate of increase, how much had he Jan. 1, 1900? Jan. 1, 1904?

436. Infinite Geometrical Series.

When $r < 1$, every term of a geometrical series is numerically smaller than the preceding term, as in $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$ It is evident that the terms *decrease* in value as the number of terms *increases*, and that as more and more terms are included, the nth term, ar^{n-1}, becomes more and more nearly equal to zero. Hence, by taking a sufficient number of terms, we can make the nth term as small as we please, although it can never become exactly zero.

Formula II, $$s = \frac{ar^n - a}{r - 1},$$

may be written $$s = \frac{a - ar^n}{1 - r}$$

$$= \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \qquad \text{(Why?)}$$

As we have seen, by taking a sufficient number of terms, we can make ar^n, and consequently the fraction $\frac{ar^n}{1 - r}$, as small as we please; the greater the number of terms, the nearer the value of this fraction approaches 0, and consequently the sum of the terms approaches $\frac{a}{1 - r}$.

And since we can take as many terms as we please, $\frac{ar^n}{1 - r}$ can be made to become less in absolute value than any positive number that may be assigned; that is, the difference between the fraction and zero will become less than any value that can be expressed by any finite number, however small. We therefore neglect $\frac{ar^n}{1 - r}$ and *call*

$$s = \frac{a}{1 - r}$$

the sum of an unlimited (*i.e.*, *infinite*) number of terms. The derivation of this formula should be reviewed in connection with the next chapter, Art. 454.

1. Find the sum of the series $1, \frac{1}{2}, \frac{1}{4}, \cdots$.

Here $a = 1, \quad r = \frac{1}{2}$.

$$\therefore s = \frac{a}{1-r} = \frac{1}{1-\frac{1}{2}} = 2.$$

The sum really is $2 - \frac{ar^n}{1-r}$; and since this neglected fraction, however small, is somewhat greater than absolute 0, the required sum is just that much less than 2. Since this difference can be made less than any number that may be assigned, however small, we are not able to give it any other value than zero, and it is therefore ignored.

2. Show that $.\dot{4}\dot{5} = \frac{5}{11}$.

$$.\dot{4}\dot{5} = .454545\cdots = .45 + .0045 + .000045 + \cdots.$$

$$\therefore a = .45, \quad r = .01.$$

$$\therefore .\dot{4}\dot{5} = s = \frac{.45}{1-.01} = \frac{45}{99} = \frac{5}{11}.$$

Find the sum of the following:

3. $1 + \frac{1}{3} + \frac{1}{9} + \cdots$.

4. $1 + \frac{1}{10} + \frac{1}{100} + \cdots$.

5. $2 + 1 + \frac{1}{2} + \cdots$.

6. $1 - \frac{1}{2} + \frac{1}{4} - \cdots$.

7. $1 + .1 + .01 + \cdots$.

8. $1 - .4 + .16 - \cdots$.

9. $1 + a + a^2 + \cdots$, when $a < 1$.

10. $2 + \frac{2}{a} + \frac{2}{a^2} + \cdots$, when $a > 1$.

Find the value of:

11. $.\dot{6}\dot{3}$.

12. $.\dot{1}4857\dot{1}$.

13. $.\dot{6}15384\dot{6}$.

14. $1.94444\cdots$.

15. $.122323\cdots$.

16. $3.185185\cdots$.

HARMONIC PROGRESSION

437. 1. What kind of series is $2, 4, 6, 8, \cdots$? Taking the *reciprocal* of each term, we have $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots$. Is this an arithmetical series? Has it a common difference?

2. Is $4, \frac{4}{3}, \frac{4}{5}, \frac{4}{7}, \cdots$ an arithmetical progression? Taking the reciprocal of each term, we have $\frac{1}{4}, \frac{3}{4}, \frac{5}{4}, \frac{7}{4}, \cdots$. What kind of series is this?

438. A series of numbers, the reciprocals of which are in arithmetical progression, is called a **Harmonic Progression,** or a **Harmonic Series.**

Thus, $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots$ is a harmonic progression, since the reciprocals, $1, 2, 3, 4, \cdots$ are in arithmetical progression.

If a violin string be made to vibrate in parts, *i.e.*, halves, thirds, quarters, etc., notes called *harmonics* are produced. The relative lengths of vibrating parts are expressed numerically by the reciprocals of the natural numbers.

1. Write the harmonic series corresponding to the arithmetical progressions $3, 5, 7, \cdots$, and $a, a+d, a+2d, \cdots$.

2. Find the 10th term of the harmonic progression $8, 4, \frac{8}{3}, \cdots$.

The corresponding arithmetical series is $\frac{1}{8}, \frac{1}{4}, \frac{3}{8}, \cdots$.

Here $a = \frac{1}{8}, \ d = \frac{1}{8}$.

$$\therefore l = a + (n-1)d, \text{ or } l = \tfrac{1}{8} + 9 \times \tfrac{1}{8} = \tfrac{10}{8}.$$

$\therefore$ the 10th term is $\frac{8}{10}$.

3. Find the 9th term of the series $1, \frac{1}{3}, \frac{1}{5}, \cdots$.

4. Find the 10th term of the series $\frac{1}{a-1}, \frac{1}{a}, \frac{1}{a+1}, \cdots$.

439. If H is the harmonic mean between a and b, then, by definition, $\frac{1}{a}, \frac{1}{H}, \frac{1}{b}$ are in arithmetical progression. Hence

$$\frac{1}{H} - \frac{1}{a} = \frac{1}{b} - \frac{1}{H}.$$

$$\therefore H = \frac{2ab}{a+b}.$$

1. Insert a harmonic mean between 3 and 7.

$$H = \frac{2 \times (3 \times 7)}{3+7} = \frac{42}{10}.$$

2. Insert 3 harmonic means between $\frac{1}{2}$ and $\frac{1}{10}$.

$$l = a + (n-1)d, \text{ or } 10 = 2 + 4d.$$

$$\therefore d = 2.$$

Hence, the arithmetical series is 2, 4, 6, 8, 10, and the three required means are $\frac{1}{4}, \frac{1}{6}$, and $\frac{1}{8}$.

3. Find the harmonic mean between 4 and -3. Between $\frac{1}{6}$ and $\frac{1}{8}$.

4. Insert 2 harmonic means between 5 and 2. Between m and n.

5. Find the 7th term of $\frac{1}{2}, \frac{1}{4}, \frac{1}{6}, \frac{1}{8}, \cdots$. The nth term.

6. Insert 4 harmonic means between .5 and -7.

7. Write the first 6 terms of a harmonic progression whose 3d and 4th terms are 2 and $1\frac{1}{3}$.

8. Find the harmonic mean between $m - n$ and $m + n$.

9. What are the two numbers whose harmonic mean is 3 and whose arithmetical mean is 4?

10. If m, n, r, are in harmonic progression, prove that $r(m - n) = m(n - r)$.

11. If the harmonic mean between m and n is r, prove that

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{r - m} + \frac{1}{r - n}.$$

12. The first two terms of a harmonic series are 5 and 6. What term will be 30?

13. The arithmetical mean between two numbers is 3, and the harmonic mean is $\frac{8}{3}$. What are the numbers?

14. Show that the geometrical mean of any two numbers is also the geometrical mean between their arithmetical and harmonic means.

15. What number must be added to each of three given numbers a, b, c, that the three results may be in harmonic progression?

16. If a, b, and c are in harmonic progression, prove that

$$\frac{b + c}{a}, \quad \frac{c + a}{b}, \quad \frac{a + b}{c}$$

are in arithmetical progression.

LIMITS, INFINITY, INDETERMINATE FRACTIONS

440. As already stated (Art. 409), a number that may have *different values* (usually an indefinite number of them) in the same expression or equation is called a **Variable**; on the other hand, a number whose value is fixed is called a **Constant**.

It is customary to represent variables by the last letters of the alphabet, and constants by the first letters.

441. It has also been stated (Art. 410) that one variable is said to be a **Function** of another when they are so related that the value of the first depends upon that of the second.

Thus, a variable y is a function of a variable x, if to every value that is given to x there corresponds a definite value of y. Under what conditions is x a function of y?

In like manner, the algebraic expression $x^2 + x + 1$ is said to be a function of x, since its value depends upon that of x. If the value of an expression depends upon the values of x and y, the expression is said to be a function of these variables.

442. For brevity, the symbol $f(x)$, read "function of x," is commonly used to denote a function of x.

Thus, any algebraic expression containing x, as $x^2 - 3x + 2$, may be conveniently denoted by $f(x)$. If two different functions of x occur in the same operation, they may be distinguished as $f(x)$ and $F(x)$.

443. To denote what a function becomes when particular values are substituted for the variable involved, we substitute the same value for the variable in the functional symbol.

Thus, if $f(x) \equiv x^2 - 2x + 1$, then $f(a)$ means that function when x is replaced by a, that is, $f(a) \equiv a^2 - 2a + 1$. Again, if $f(x) \equiv x^2 - 5x + 6$, then $f(2) = 4 - 10 + 6 = 0$.

LIMITS

444. If a variable x as it changes its value approaches nearer and nearer some constant a, so that the difference in absolute value between x and a can become and remain less than any assignable positive number, then the constant a is called the **Limit** of the variable x.

For example, let the line AB be bisected at C, and CB at D, and DB at E, and so on, and suppose that a point P moves from A toward B under the condition that it shall move first from A to C, half the distance from A to B; next to D, half the distance from C to B; then to E, half the distance from D to B; and so on indefinitely.

Obviously, as P thus moves from A to B, its variable distance from A, represented by x, becomes more and more nearly equal to the constant distance AB, represented by a. Hence it is seen that by continuing indefinitely the motion of P under the given law, the difference between AP and AB may be made as small as we please, that is, less than any assignable distance, yet distinct from 0, since half of any remaining distance is greater than 0. In this case, AB, or a, is the *limit* of the variable AP, or x.

It should be observed that if the point P simply slides along the line AB, without conforming to the above law of movement, then although the difference between AP and AB can *become* smaller than any assignable distance, it does not *remain* smaller, since P will reach and pass through B in its movement, after which the distance between AP and AB will *increase*. In such case, AB is not the limit of AP.

445. The symbol $\doteq$, read "approaches as its limit," is placed between a variable and a constant to denote that the former approaches the latter as its limit.

Thus, $x \doteq a$ means that "x approaches a as its limit."

446. It follows at once from the definition of a limit, that *if* $x \doteq a$, *then* $a - x$ *is a variable whose limit is* 0; *that is*, $(a - x) \doteq 0$.

Returning to the illustration given in Art. 444, it is evident that the distance from the moving point P to B is also a variable which, as AP *increases* toward the limit AB, *decreases* toward the limit 0. But the variable distance from P to B is the difference between a and x, so that $(a - x) \doteq 0$, as $x \doteq a$.

447. For any value of x except 0 the fraction $\frac{1}{x}$ has a corresponding value which is readily computed; but to find the value of the fraction when $x = 0$ involves a meaningless operation, since there is no number, n, such that $0 \times n = 1$, as required in this case. In investigating the value of the fraction as x *decreases* in absolute value, we are led to certain mathematical concepts with which we are as yet unacquainted.

448. In considering the series of fractions $\frac{1}{1}, \frac{1}{.1}, \frac{1}{.01}, \frac{1}{.001}, \frac{1}{.0001}, \ldots$, to which the fraction $\frac{1}{x}$ gives rise by giving to x first the value 1, and then in succession a value one tenth of that which it has in each preceding case, it is apparent that as the value of x becomes smaller and smaller, the value of $\frac{1}{x}$ becomes larger and larger and may be made to increase indefinitely by taking x sufficiently small, but not 0. That is to say, as x decreases in absolute value, $\frac{1}{x}$ may be made to exceed a *given* positive number, say 1,000,000, by taking x sufficiently small, namely, less than .000001 in the supposed case. Similarly, by taking x smaller and smaller indefinitely, the value of $\frac{1}{x}$ may be made to exceed *any* assigned positive number, however great.

INFINITY

449. If a variable, x, can be made to become and remain greater in absolute value than any positive number that may be assigned, however great, the variable is said to increase *without limit* (Latin, *in-finitus*, "without bound"). For the sake of brevity, we say that "x becomes *infinite*," or that "x approaches *infinity*," expressions which mean only that "x increases without limit."

450. The symbol for the limit of such a variable is ∞, read "*infinity*." The student should understand clearly that this

symbol does not denote a *definite numerical value*. As used in the expression $x \doteq \infty$, it means that "x increases in absolute value beyond any assignable positive number."

451. In contradistinction to *Infinites*, the numbers hitherto used are called *Finite Numbers*. From the nature of infinites it is evident that they do not obey the ordinary laws of finite numbers, as their efficacy in the following operational combinations shows:

$$\infty + a = \infty,$$

$$\infty - a = \infty,$$

$$a \times \infty = \infty \ (a \neq 0),$$

$$\frac{\infty}{a} = \infty.$$

452. It should be observed that the definition of infinites implies that *fixed* values cannot be assigned to them. When a variable x becomes infinite, it is said to be indeterminately great. We can affirm, and only affirm, that x is some number in the class of unassignably great or infinite numbers. From the preceding it is seen that infinity is not identically equal to infinity. Is a value that renders both members of an equation infinite a root of that equation? Why not?

453. It follows from Art. 448 that, if a is a finite number, and not 0, the value of the fraction $\frac{a}{x}$ increases without limit, as $x \doteq 0$. That is, *if the numerator is a finite number and not* 0, *and the denominator approaches* 0, *the absolute value of the fraction will become infinite;* or expressed in symbols, $\frac{a}{x} \doteq \infty$, as $x \doteq 0$.

As we have already seen, the form $\frac{a}{0}$ is meaningless as a division operation, but the expression $\frac{a}{0} = \infty$ is often used as an abbreviated notation for "when x approaches 0 as its limit, then $\frac{a}{x}$ increases without limit."

454. In considering the fraction $\frac{a}{x}$, where a is a finite number, and not 0, and x is a variable, we notice that by taking x sufficiently large the absolute value of $\frac{a}{x}$ may be made as small as we please, or brought as near 0 as we please. Hence $\frac{a}{x}$ is said to approach the limit 0 as x increases indefinitely. That is, *if the numerator of a fraction is a finite number and not 0, and the denominator becomes infinite, the value of the fraction will approach* 0; or expressed in symbols, $\frac{a}{x} \doteq 0$, as $x \doteq \infty$.

INDETERMINATE FRACTIONS

455. Certain fractional expressions are met with in algebra that for particular values of the variables involved assume what are known as *indeterminate forms*.

456. The Form $\frac{0}{0}$.

If we consider the fraction $\frac{x^2 - a^2}{x - a}$, which is a function of x, we find that the fraction has a meaning for all values of x, except $x = a$. If we place $x = a$, the fraction assumes the form $\frac{0}{0}$, which may be equal to *any finite number whatever*. For, by definition of a fraction, $\frac{0}{0}$ represents a number which, when multiplied by 0, produces 0. But any finite number multiplied by 0 gives 0. Hence it is seen that the fraction does not have any definite value for $x = a$, and is therefore said to be **Indeterminate**.

Although the fraction is indeterminate, we may show that it approaches a limit, as $x \doteq a$, and then proceed to indicate a method for determining the value which it approaches as its limit. This value is called the **Limiting Value** of the fraction.

Now, as long as $x \neq a$, $\frac{x^2 - a^2}{x - a} \equiv (x + a)\frac{x - a}{x - a} \equiv x + a$, and, obviously, as $x \doteq a$, $(x + a)\frac{x - a}{x - a} \doteq (a + a)$, or $2a$.

That is to say, if a fraction containing x assumes an indeterminate form for $x = a$, say, we determine the limit which it approaches as $x \doteq a$.

This leads us to *define* the value of an indeterminate fraction to mean the value which it approaches as a limit, as the variable approaches the particular value which renders the fraction indeterminate.

457. The Form $\frac{\infty}{\infty}$.

So, too, the fraction $\frac{\infty}{\infty}$ may have any value, and so is *indeterminate*. For, obviously, any finite number multiplied by a number that increases without limit will give a number increasing without limit; that is, any finite number multiplied by ∞ gives ∞.

1. Find the limiting value of $\frac{x^2-4}{x-2}$, when $x \doteq 2$.

For $x = 2$, $\quad \frac{x^2-4}{x-2}$ becomes $\frac{0}{0}$.

But $\quad \frac{x^2-4}{x-2} = (x+2)\frac{x-2}{x-2}$,

and, as $x \doteq 2$, $\quad (x+2)\frac{x-2}{x-2} \doteq x+2$, or 4.

2. Find the limiting value of $\frac{x-1}{x+1}$, when x increases without limit; that is, when $x \doteq \infty$.

As $x \doteq \infty$, $\quad \frac{x-1}{x+1}$ becomes $\frac{\infty}{\infty}$.

But $\quad \frac{x-1}{x+1} \equiv \frac{1-\frac{1}{x}}{1+\frac{1}{x}}$, $\quad (x \neq 0)$

and, since $\frac{1}{x} \doteq 0$, as $x \doteq \infty$, it is evident that, as $x \doteq \infty$,

$$\frac{1-\frac{1}{x}}{1+\frac{1}{x}} \doteq \frac{1}{1}, \text{ or } 1.$$

3. Find the limiting value of $\frac{1-x^3}{1-x}$, when $x \doteq 1$.

For $x = 1$, $\frac{1-x^3}{1-x}$ becomes $\frac{0}{0}$.

But $\frac{1-x^3}{1-x} = 1 + x + x^2$,

and, as $x \doteq 1$, $1 + x + x^2 \doteq 3$.

It is noticed that the limiting value of the fraction is found by removing the common factors in x from the numerator and denominator, and substituting in the result the value which x approaches.

Find the limiting values of each of the following fractions:

4. $\frac{x^2-9}{x+3}$, when $x \doteq \infty$.

5. $\frac{x^3+1}{x^2-1}$, when $x \doteq -1$.

6. $\frac{2x+1}{5x+3}$, when $x \doteq \infty$.

7. $\frac{x^2-5x+6}{x^2+2x-8}$, when $x \doteq 2$.

8. $\frac{x^5-1}{x^3-1}$, when $x \doteq 1$.

9. $\frac{4x-3}{2x+1}$, when $x \doteq \infty$.

10. $\frac{7x^2-2x-5}{2x^2-7x+3}$, when $x \doteq \infty$.

11. $\frac{x-2}{\sqrt{2x-2}}$, when $x \doteq 2$.

12. $\frac{ax+b}{bx+a}$, when $x \doteq \infty$.

13. $\frac{x^2-2x-3}{x-3}$, when $x \doteq 3$.

14. $\frac{x^2-4x-5}{x+1}$, when $x \doteq -1$.

15. $\frac{x^2+3x-4}{x(x-1)}$, when $x \doteq 1$.

16. $\frac{x^2-x-6}{x^2+7x+10}$, when $x \doteq -2$.

17. $\frac{x^2+5x-6}{x^2-1}$, when $x \doteq 1$.

18. $\frac{4x^2-3x+7}{x^2-5x+1}$, when $x \doteq \infty$.

19. $\frac{x^2+2x-24}{x^2-7x+12}$, when $x \doteq 4$.

20. $\frac{x^3+b^3}{x^2-b^2}$, when $x \doteq -b$.

21. $\frac{x^2-4ax+3a^2}{x^2-a^2}$, when $x \doteq a$.

DISCUSSION OF THE SOLUTIONS OF PROBLEMS

458. The significance of indeterminate and infinite results may be further illustrated by the solution of certain problems in which they occur.

Indeterminate Results

459. A father is 45 years old, and his son 15. After how many years will the father be 30 years older than his son?

Let $x =$ the required number of years.

Then $45 + x = 15 + x + 30,$

or $45 + x \equiv 45 + x,$

an *identity* which is satisfied by *any finite value of* x.

Completing the solution in the usual way,

$$x - x = 45 - 45,$$

or $$(1 - 1)x = 45 - 45.$$

$$\therefore x = \frac{45 - 45}{1 - 1} = \frac{0}{0}.$$

The expression $\frac{0}{0}$ is interpreted to mean that the conditions of the problem will be satisfied when x equals any finite number. This result is evidently true, for from the problem the father will at any time be 30 years older than his son.

Infinite Results

460. A cistern can be filled by one pipe in 4 hours and by another in 6 hours, and it can be emptied by a third in $\frac{1}{t}$ hours. Were the cistern empty and all the pipes opened together, in what time would it be filled? In what time, if $\frac{1}{t} = \frac{12}{5}$?

Let $x =$ the number of hours required.

Then $\frac{1}{x} =$ the part filled by all in 1 hour.

Now $\frac{1}{4} =$ the part filled by the first in 1 hour,

$\frac{1}{6} =$ the part filled by the second in 1 hour,

and t, or $\frac{5}{12} =$ the part emptied by the third in 1 hour.

By the conditions, $$\frac{1}{4} + \frac{1}{6} - t = \frac{1}{x}.$$

$$\therefore \frac{1}{x} = \frac{5}{12} - t. \qquad (1)$$

$$\therefore x = \frac{1}{\frac{5}{12} - t}. \qquad (2)$$

Obviously, if we let t approach $\frac{5}{12}$, the denominator of the second member will approach 0, and the value of x will become *infinite*. This result means that the cistern will never be filled under the conditions assumed.

If $\frac{1}{t} = \frac{12}{5}$, then $t = \frac{5}{12}$, and (1) becomes

$$\frac{1}{x} = 0, \tag{3}$$

and (2) becomes

$$x = \frac{1}{0}. \tag{4}$$

It is seen from (3) that there is no *finite* value for x such that $\frac{1}{x} = 0$. This indicates that the cistern will never be filled, the part *filled* by all in 1 hour being equal to 0. But if the solution is completed as in (4), the denominator of the second member is exactly 0, and no solution exists, the *equation* being impossible. It does not necessarily follow that the *problem* is impossible, but the expression for the value of x having no literal meaning, the equation does not enable us to affirm anything of the problem.

From the numerical data given in the problem, it is evident that the part filled by the two pipes in 1 hour is just the same as the part emptied by the third in the same time, and hence that the cistern will never be filled.

PROBLEM OF THE COURIERS

461. Two couriers, A and B, travel in the direction M to N, at the respective rates of a miles and b miles an hour. A reaches P at a time t hours before B reaches Q, which is d miles beyond P. After how many hours will the couriers be together?

Suppose them to be together at R, a point beyond Q, and let x = the number of hours traveled after B reaches Q.

Since B travels bx miles in x hours, the distance from Q to R is bx miles; since A travels $a(t+x)$ miles in $t+x$ hours, the distance from P to R is $a(t+x)$ miles. Hence, the distance from Q to R is $a(t+x)-d$ miles, and we have

$$a(t+x) - d = bx. \qquad \text{(Why?)}$$

$$\therefore\ x = \frac{d-at}{a-b}.$$

I. Positive Result. If either $d > at$ and $a > b$, or $d < at$ and $a < b$, then x is positive. A positive result means that the problem is possible under the conditions assumed.

When $d > at$ and $a > b$, $d > at$ indicates that A has not yet reached Q when B arrives at that point, while $a > b$ shows that A is traveling faster than B, and will therefore overtake him at some point beyond Q.

When $d < at$ and $a < b$, $d < at$ indicates that A has passed Q when B arrives, while $a < b$ shows that B is traveling faster than A, and will therefore overtake the latter at some point beyond Q. In either case the meeting takes place at some point beyond Q.

II. Negative Result. If either $d > at$ and $a < b$, or $d < at$ and $a > b$, then x is negative. A negative result means that the supposition that the couriers are together at the right of Q is incorrect. But since a positive result means that the couriers are together at some point to the right of Q, the negative result may be interpreted to mean that, under the assumed conditions, they were together at some point to the left of Q. It has also been seen that the statement of the problem may be so changed as to make the result applicable (Art. 215).

By the reasoning used in I, when $d > at$ and $a < b$, $d > at$ indicates that A has not yet reached Q when B arrives at that point, while $a < b$ shows that B is traveling faster than A, and must therefore have passed A at some point to the left of Q.

When $d < at$ and $a > b$, $d < at$ indicates that A has passed Q when B arrives, while $a > b$ shows that A is traveling faster than B, and must therefore have overtaken the latter at some point to the left of Q. In either case, under the conditions, the couriers could only be together at some point to the left of Q.

III. Zero Result. If $d = at$ and $a \neq b$, then $x = 0$. This result means that they meet at Q, both arriving at that point at the same time. Show how this interpretation corresponds with the assumptions made in the problem.

IV. Indeterminate Result. If $d = at$ and $a = b$, then the value of x is *indeterminate*. This result means that they are always together, every point of the road being regarded as a place of meeting. This is evident from the problem, since if $d = at$, they arrive together at Q, and their rates being equal, they must be together at all times.

V. Infinite Result. If $d \neq at$ and $a = b$, then the denominator approaches 0, and x becomes *infinite*. This result means that the couriers will never be together, that is, that a meeting is impossible under the conditions assumed. When $d \neq at$ and $a = b$, how does it appear from the problem that a meeting is impossible?

Solve the following problems and interpret the results:

1. What number added to both numerator and denominator of the fraction $\frac{3}{8}$ will make the resulting fraction equal to $\frac{1}{2}$?

2. The sum of the first and third of three consecutive even numbers is equal to twice the second. Find the numbers.

3. A father is 30 years older than his son, and the sum of their ages is 30 years less than twice the father's age. Find the son's age.

4. What number added to the denominators of the fractions $\frac{a}{b}$ and $\frac{c}{d}$, will make the resulting fractions equal?

5. A dealer has two kinds of wine worth a and b dollars a gallon, respectively. How many gallons of each must he take to make a mixture of n gallons worth c dollars a gallon?

6. A and B go on a trip, A having \$500, B \$150. A's expenses are double B's, yet he has left three times as much as B has left. Required the expenses of each.

7. A cistern can be filled by one pipe in m minutes, and by another in n minutes, and it can be emptied by a third in p minutes. Were the cistern empty and all the pipes opened together, in what time would it be filled? Discuss the problem for $\frac{1}{p} < \frac{1}{m} + \frac{1}{n}$, and for $\frac{1}{p} = \frac{1}{m} + \frac{1}{n}$.

8. Two bicyclists, A and B, start at the same time from two places d miles apart, and ride in the same direction. B travels at the rate of a miles an hour and A travels c times as fast. Where will A overtake B?

9. A man drives from his home to his station at the rate of a miles an hour, but on arriving at the station finds that his train has left c hours before. If the distance from his house to the station is d miles, at what rate should he have driven to catch his train? Discuss the problem for (1) $c = 0$; (2) $c = \frac{d}{a}$; (3) $c = \frac{-d}{a}$.

LOGARITHMS

462. Examine the following series of numbers:

Exponents: 0, 1, 2, 3, 4, 5, 6, 7, 8, ... (1)

Powers: $2^0, 2^1, 2^2, 2^3, 2^4, 2^5, 2^6, 2^7, 2^8, \ldots$. (2)

$1, 2, 4, 8, 16, 32, 64, 128, 256, \ldots$. (3)

It is seen that each number in (3) is a *power* of the fixed number 2, and that the corresponding number in (1) is the *exponent* of that power.

Since $8 = 2^3$ and $16 = 2^4$, we have

$$8 \times 16 = 2^3 \times 2^4 = 2^7 = 128.$$

Similarly, $4 \times 64 = 2^2 \times 2^6 = 2^8 = 256.$

It is thus apparent that these products are found by *adding the exponents* of the fixed number 2, and taking the corresponding power of 2 from (3).

Show that quotients may be found by *subtracting the exponents* of 2 and taking the corresponding power of 2 from (3).

We now come to the consideration of a scheme by which the powers of any number may be arranged in a table and products, quotients, powers, and roots be found by the addition, subtraction, multiplication, and division of exponents.

463. The *exponent* of the power to which a *fixed number* must be raised to produce *a given number* is called the **Logarithm** of that number.

When 10 is the fixed number, and 100 the given number, the exponent 2 is the logarithm, because $10^2 = 100$; if 1000 were the given number, its logarithm would be 3, because $10^3 = 1000$. Since $10^4 = 10{,}000$, what is the logarithm of 10,000?

464. The *fixed number* whose exponent is the logarithm of a given number is called the **Base**. Any positive number except 1 may be taken as a base for computing logarithms. The number *one* is excepted because its powers are no greater than itself.

465. When the base is 10, the logarithms are called **Common** or **Brigg's Logarithms**. These are the only ones used for numerical computations. If no base is mentioned, the base 10 is understood. In this system all numbers are regarded as *powers of ten.*

Thus, $100 = 10^2$; $20 = 10^{1.3010\cdots}$; $124 = 10^{2.0934\cdots}$

In these expressions 10 is the base, and the exponents are the logarithms. Thus, 2 is the logarithm of 100 to the base 10, etc. Using the logarithmic form, we have

$$\log_{10} 100 = 2, \text{ or simply } \log 100 = 2;$$

$$\log\ 20 = 1.3010+; \quad \log 124 = 2.0934+.$$

Although the exponent indicating the power to which 10 must be raised cannot be expressed exactly when it is not an integral power of 10, it can be found to any required degree of accuracy by using approximate values. The laws which were found to apply to rational exponents apply also to irrational exponents.

466. In general, when a is the base, x the exponent, and n the given number, we have $a^x = n$, in which x is the logarithm of n to the base a. It is written $\log_a n = x$.

In theoretical investigation use is made of what is known as **Natural Logarithms**. The base of this system is 2.7182818 ···, denoted by the letter e. When 10 is the base it is not usually expressed.

Logarithms were invented about the year 1614 by a Scotchman, John Napier. The base 10 was suggested by an Englishman, Henry Briggs, a contemporary of Napier. The chief advantage of the base 10 lies in the fact that it is also the base of our decimal system, and that as a consequence from the known logarithm of a given number there may be derived by inspection the logarithms of all numbers differing from the given number only in the position of the decimal point.

467. It follows from the preceding that

$\because 10^0 = 1, \quad \therefore \log 1 = 0;$

$\because 10^1 = 10, \quad \therefore \log 10 = 1;$

$\because 10^2 = 100, \quad \therefore \log 100 = 2;$

$\because 10^3 = 1000, \quad \therefore \log 1000 = 3;$

$\because 10^{-1} = \frac{1}{10}, \quad \therefore \log .1 = -1;$

$\because 10^{-2} = \frac{1}{100}, \quad \therefore \log .01 = -2,$ and so on.

468. It is therefore evident that the logarithm of any number between 1 and 10 is greater than 0 and less than 1, that is, it is 0 plus a fraction; that the logarithm of any number between 10 and 100 is greater than 1 and less than 2, that is, it is 1 plus a fraction; and so on. Thus, the logarithm of 9 has been found to be 0.9542 +, and that of 25 to be 1.3979 +.

It is also evident that the logarithm of a number greater than 1 is positive, while that of a number less than 1 and greater than 0 is negative.

469. Only exact powers of 10 have *integral* logarithms. All other numbers have logarithms that cannot be expressed exactly. They are 0 or an integer, plus a decimal that does not terminate.

Only the numbers between 1 and 10 have logarithms that are wholly fractional — that are 0 plus a decimal.

470. The decimal part of a logarithm is called the **Mantissa**; the integral part is called the **Characteristic.**

(a) $\log 16 = 1.2041$; (b) $\log 100 = 2$; (c) $\log 4 = 0.6021$.

In (a), .2041 is the mantissa, 1 the characteristic ; in (b), there is no mantissa — the logarithm is integral, 100 being a power of 10 ; in (c), the characteristic is 0. Why ?

471. Since 6043 lies between 10^3 and 10^4, that is, between 1000 and 10,000, its logarithm lies between 3 and 4. By calculation it has been found to be 3.7812, the integral part, 3, being the characteristic, and the decimal part, .7812, the mantissa.

Carefully consider the following examples:

$10^{3.7812} = 6043$; $\therefore \log 6043 = 3.7812$.

$10^{2.7812} = 604.3$; $\therefore \log 604.3 = 2.7812$. (Why ?)

$10^{1.7812} = 60.43$; $\therefore \log 60.43 = 1.7812$.

$10^{0.7812} = 6.043$; $\therefore \log 6.043 = 0.7812$.

$10^{-1+.7812} = .6043$; $\therefore \log .6043 = \bar{1}.7812$, or $.7812 - 1$.

$10^{-2+.7812} = .06043$; $\therefore \log .06043 = \bar{2}.7812$, or $.7812 - 2$.

$10^{-3+.7812} = .006043$; $\therefore \log .006043 = \bar{3}.7812$, or $.7812 - 3$.

It is evident from these examples that:

1. If a number is greater than 1, the characteristic of its logarithm is positive and is 1 less than the number of integral places in the number.

Thus, if a characteristic is 3, we know at once that the number contains 4 integral places.

2. If a positive number is less than 1, the characteristic of its logarithm is negative, and is numerically 1 greater than the number of ciphers between the decimal point and the first significant figure. When the minus sign is written over the characteristic, it denotes that it alone is negative.

Thus, if a characteristic is -3, we know at once that the decimal has 2 ciphers following the decimal point. *The mantissa is always positive.*

3. If numbers are expressed by the same figures in the same order, their *logarithms differ only in their characteristics, the decimal parts remaining the same.*

This follows from the fact that moving the decimal point is equivalent to dividing or multiplying the number by a power of 10, which increases or diminishes the logarithm by the exponent of that power; and this exponent, being integral, affects only the characteristic.

The foregoing explains why tables of logarithms contain only the *mantissas*. The characteristic can always be found by inspection.

472. 1. What is the characteristic of the logarithm of a number of 2 integral places? Of 3? Of 5? Of 10?

2. What is the characteristic of the logarithm of .2? Of .05? Of .007?

3. If the logarithm of 6742 is 3.8288, what is the logarithm of 67,420? Of 674,200? Of 674.2? Of 67.42?

Since only positive numbers are employed as bases in any system of logarithms (Art. 464), negative numbers are not considered as having logarithms, but operations involving negative numbers are performed as if the numbers were positive, and the proper sign prefixed to the result.

473. A table of four-place logarithms is given on pages 406–407. This table contains the mantissas of the logarithms of all numbers under 1000, the characteristic and decimal point being omitted. The given numbers from 10 to 99 are found in the columns headed "N"; the mantissas are found in the other columns.

In this table the mantissas are correct to three decimal places. The fourth figure is also correct unless the fifth is 5 or more, in which case the fourth is increased by 1, just as we write \$ 1.38 for \$ 1.375. Four-place tables are sufficiently exact for all ordinary calculations; for work requiring greater accuracy more extended tables are used.

474. To find the Logarithm of a Number.

The mantissa of the logarithm of any integral number from 10 to 100 can be found opposite the given number on pages 406–407. The mantissas of the numbers 1, 2, 3, ···, 9, are the same as those of 10, 20, 30, ···, 90 (Art. 471, 3), so that the table may be conveniently begun with the number 10. The following examples will illustrate the method of finding the logarithms of any other numbers.

1. Find the logarithm of 678, of 67.8, of .678, and of .0678.

On the second page of the table, the first two figures of 678 are found in the left-hand column headed "N." On a line with these and in the column headed by the third figure 8, we find 8312. With the decimal point prefixed, the result .8312 is the mantissa of the log 678.

Since the given number has three integral figures, the characteristic is 2 (Art. 471, 1). $\therefore \log 678 = 2.8312.$

Since dividing a number by 10 decreases its logarithm by 1 (Art. 471, 3), we know that

$$\log 67.8 = 1.8312,$$

$$\log 6.78 = 0.8312,$$

$$\log .678 = \bar{1}.8312 = .8312 - 1,$$

$$\log .0678 = \bar{2}.8312 = .8312 - 2.$$

The mantissas are found in the table, but the characteristics must be determined by an inspection of the given number; that is, by application of the principles stated in Art. 471, 1, 2.

2. Find the logarithm of 5943. Also the logarithm of 59.43 and that of 594,300.

If the given number is expressed by four or more figures, it is necessary to determine the logarithms of the numbers of three figures which are respectively less than and greater than the given number.

In the case of 5943, the characteristic is 3. The mantissa of 594 is found in the line with 59 and under 4; it is .7738. In the same way the mantissa of 595 is found to be .7745.

$$\therefore \log 5940 = 3.7738, \text{ and } \log 5950 = 3.7745.$$

The logarithm of 5943 lies between these two logarithms, whose difference is .0007.

But 5950 exceeds 5940 by 10, and 5943 exceeds 5940 by 3; that is, 5943 exceeds 5940 by $\frac{3}{10}$ of the difference between 5940 and 5950. Hence the log 5943 must exceed log 5940 by approximately $\frac{3}{10}$ of .0007.

$$\therefore \log 5943 = 3.7738 + \tfrac{3}{10} \text{ of } .0007$$

$$= 3.7738 + .0002 = 3.7740.$$

Dividing a number by 100 decreases its logarithm by 2, hence

$$\log 59.43 = 1.7740.$$

Since multiplying a number by 10 increases its logarithm by 1, and multiplying by 100 increases the logarithm by 2,

$$\therefore \log 594300 = 5.7740.$$

In adding differences the fifth figure is ignored unless it is 5 or more, in which case it is dropped and the *fourth figure* is increased by 1.

3. Find the logarithm of 0.0063518.

Since there are two ciphers between the decimal point and 6, the characteristic is -3. Proceeding to find the mantissa as in the preceding example, we have

$$\log .0063500 = \bar{3}.8028,$$
$$\log .0063600 = \bar{3}.8035.$$
$$\text{Difference} = .0007.$$

But .0063518 exceeds .0063500 by $\frac{18}{100}$ of the difference between .0063500 and .0063600.

$$\therefore \log 0.0063518 = \bar{3}.8028 + \tfrac{18}{100} \text{ of } .0007$$
$$= \bar{3}.8028 + .0001$$
$$= \bar{3}.8029, \text{ or } 0.8029 - 3.$$

475. To find the logarithm of a number of four or more significant figures requires a process called **Interpolation,** as shown in the two preceding examples. In this process the assumption is made that the numbers vary as their logarithms, and hence that changes made in the numbers produce proportional changes in the logarithms. But this is only approximately true, since the numbers form a geometric series and the logarithms an arithmetical series. However, the assumption is true for relatively *small differences* in the numbers; the error will not in general affect the fourth figure in the mantissa.

Find the logarithm of:

1. 18.	6. 18.03.	11. 3.1416.
2. 216.	7. 37.25.	12. .00021.
3. 750.	8. 582.7.	13. .23068.
4. 1254.	9. 3.146.	14. .03862.
5. 1803.	10. .7854.	15. .00631.

476. To find the Antilogarithm.

The number that corresponds to a given logarithm is called the **Antilogarithm.** The process of finding this when a logarithm is given is the reverse of the process of finding the logarithm of a number. The location of the decimal point depends upon the characteristic of the given logarithm.

N	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

N	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

1. Find the number corresponding to the logarithm .5922.

Look for 5922 in the table on pages 406–407. In the column headed "N," and on a line with 5922, is found 39, the first two figures of the required number. The third figure, 1, is found heading the column in which 5922 stands. The three figures of the number are 391, and since the characteristic is 0, the number is 3.91. (Art. 471, 1.)

2. Find the number corresponding to the logarithm 2.9779.

Since the characteristic is 2, there are *three integral* places in the required number.

The given mantissa is not found in the table, but the next less mantissa is .9777 and the next greater .9782. Their difference is .0005. But the given mantissa exceeds the next less mantissa by .0002; that is, by $\frac{2}{5}$ of the difference between the two adjacent mantissas. Hence the antilog 2.9779 must exceed antilog 2.9777 by approximately $\frac{2}{5}$ of the difference between the numbers corresponding to the logarithms 2.9777 and 2.9782. Thus,

$$\begin{array}{rrr} \text{antilog } 2.9782 = 951 & & 2.9779 \\ \text{antilog } 2.9777 = 950 & & 2.9777 \\ \hline .0005 \quad 1 & & .0002 \end{array}$$

$$\therefore \text{antilog } 2.9779 = 950 + \tfrac{2}{5} \text{ of } 1$$
$$= 950 + .4 = 950.4.$$

3. Find the number whose logarithm is $\bar{3}.8029$.

Since the characteristic is -3, two ciphers follow the decimal point. The two adjacent mantissas are .8028 and .8035. Then

$$\begin{array}{rrr} \text{antilog } \bar{3}.8035 = .00636 & & \bar{3}.8029 \\ \text{antilog } \bar{3}.8028 = .00635 & & \bar{3}.8028 \\ \hline .0007 \quad .00001 & & .0001 \end{array}$$

$$\therefore \text{antilog } \bar{3}.8029 = .00635 + \tfrac{1}{7} \text{ of } .00001$$
$$= .00635 + .000001+$$
$$= .006351.$$

In the interpolation the additive part cannot be relied upon beyond four significant figures except when more extended tables are used. Hence in this case the result is not carried beyond the sixth decimal place.

Find the number corresponding to

4. 0.3010.	7. 2.2304.	10. 1.9501.	13. 1.0413.
5. 1.1760.	8. 3.4473.	11. 2.3432.	14. $\bar{2}.2648$.
6. 1.9822.	9. 0.8794.	12. 0.4207.	15. $\bar{3}.5910$.

PROPERTIES OF LOGARITHMS

477. Since logarithms are exponents, the laws of exponents give rise to the corresponding properties of logarithms. The laws of logarithms are established and applied in the articles which immediately follow.

478. Computations by Logarithms.

The logarithm of a product equals the sum of the logarithms of the factors.

Thus, $100 \times 1000 = 10^2 \times 10^3 = 10^5$.

$\therefore \log(100 \times 1000) = 2 + 3 = \log 100 + \log 1000$.

In general, if $\log m = x$ and $\log n = y$, we have

$$mn = 10^x \times 10^y = 10^{x+y}. \qquad \text{(Art. 290)}$$

$$\therefore \log(mn) = x + y = \log m + \log n.$$

The same reasoning holds for any other base.

1. Multiply 0.0468 by 75.

$$\log(.0468 \times 75) = \log .0468 + \log 75.$$

$$\log .0468 = 0.6702 - 2$$

$$\log 75 = 1.8751$$

$$\log .0468 + \log 75 = 2.5453 - 2$$

$\therefore$ log product $= 0.5453$.

Corresponding number $= 3.51$, the product.

When an odd number of factors are *negative*, the product is negative. Otherwise the process is the same as if all factors were positive. In practice it is less confusing if the negative characteristic is written by itself at the right as above.

Multiply the following:

2. 74 by 385.
3. 6.2 by 249.
4. 4.6 by 8.3.
5. 123 by .542.
6. .0368 by 2.15.
7. .5007 by .263.
8. .371 by 17.3.
9. .009 by 9.008.
10. 144 by .7854.
11. -0.8724 by 42.5.
12. -8×7 by -6×126.
13. .00632 by 876.3.

479. The logarithm of a quotient equals the logarithm of the dividend minus that of the divisor.

Thus, $10000 \div 100 = 10^4 \div 10^2 = 10^2$.

$\therefore \log(10000 \div 100) = 4 - 2 = \log 10000 - \log 100.$

In general, if $\log m = x$ and $\log n = y$, we have

$$m \div n = 10^x \div 10^y = 10^{x-y}. \qquad \text{(Art. 294)}$$

$$\therefore \log(m \div n) = x - y = \log m - \log n.$$

1. Divide 52.5 by 234.87.

$$\log(52.5 \div 234.87) = \log 52.5 - \log 234.87.$$

$$\log 52.5 = 1.7202$$

$$\underline{\log 234.87 = 2.3708}$$

$$\log 52.5 - \log 234.87 = \bar{1}.3494 = 0.3494 - 1.$$

Corresponding number = .2235+.

Divide the following:

2. 1875 by 25.
3. 2025 by 45.
4. 3.1416 by 31.
5. .07854 by .07.
6. 149 by 37.25.
7. 25.74 by 1.1.
8. 8 by .003764.
9. 7.06 by .07023.

480. It follows from Art. 479 that the logarithm of the reciprocal of any number m is $\log 1 - \log m$. This is called the **Cologarithm** of m. And since $\log 1 = 0$, we have

$$\operatorname{colog} m = -\log m.$$

It is customary to write $(10 - \log m) - 10$ instead of $-\log m$ in order to avoid a *negative mantissa*.

Since multiplying by the reciprocal of a number is equivalent to dividing by the number, we may find the logarithm of a *quotient* by *adding the cologarithm* of the divisor to the logarithm of the dividend.

Thus, since $\log 5 = 0.6990$, $\operatorname{colog} 5 = -0.6990$, which may be written $10 - 0.6990 - 10$, or $9.3010 - 10$. That is, the cologarithm may be found by subtracting the logarithm from 10 and then subtracting 10 from this result. The subtraction is readily performed from left to right by taking

each digit, except the last, from 9. When the last digit is 0, the one next it must be subtracted from 10. The chief advantage in using cologarithms arises from the fact that a chain of operations may be performed by the *addition* of logarithms and cologarithms.

1. Find the value of $\dfrac{312 \times 75}{7154 \times 0.19}$.

$$\begin{aligned} \log 312 &= 2.4942 \\ \log\ 75 &= 1.8751 \\ \operatorname{colog} 7154 &= 6.1455 - 10 \\ \operatorname{colog} 0.19 &= 10.7212 - 10 \\ \text{Sum} &= 21.2360 - 20 \\ &= 1.2360. \end{aligned}$$

$$\text{Antilog } 1.2360 = 17.22.$$

$$\therefore \frac{312 \times 75}{7154 \times 0.19} = 17.22.$$

Find the value to four significant places:

2. $\dfrac{213 \times 57}{7865 \times 0.179}$.

3. $\dfrac{317 \times 92}{6178 \times 0.13}$.

4. $\dfrac{8124 \times .00345}{.00069 \times 87.42}$.

5. $\dfrac{.0176 \times 68.59}{3245 \times .7246}$.

6. $\dfrac{3.24 \div 65}{6.32 \times 7832}$.

7. $\dfrac{121.6 \times 9.025}{48.3 \times 3662 \times .0856}$.

8. $\dfrac{273.2 \times 63.87}{938 \div 134.6}$.

9. $\dfrac{161.4 \times 15}{32 \times 37.1 \times 5.76}$.

10. $\dfrac{.45 \times 204 \div 18}{20 \div 400 \times .005}$.

11. $\dfrac{4.9 \times (-306) \times 48.3}{100 \times 2.9 \times .081}$.

481. The logarithm of a power of a number is equal to the logarithm of the number multiplied by the exponent of the power.

Thus, $100^3 = (10^2)^3 = 10^{2\times 3} = 10^6.$

$$\therefore \log 100^3 = 2 \times 3 = 6.$$

In general, if $\log m = x$ and a = any exponent of m, we have

$$m^a = (10^x)^a = 10^{ax}. \qquad \text{(Art. 295)}$$

$$\therefore \log m^a = ax = a \times \log m.$$

1. Find the third power of 15.

$$\log 15 = 1.1761.$$
$$3 \times \log 15 = 3.5283.$$
$$\text{Corresponding number} = 3375 = 15^3.$$

2. Find the fourth power of .18.

$$\log .18 = \bar{1}.2553 = 0.2553 - 1.$$
$$4 \times \log .18 = \quad 1.0212 - 4$$
$$= 0.0212 - 3, \text{ or } \bar{3}.0212.$$

Corresponding number = .00105.

This result differs slightly from that obtained by actual multiplication, as $.18^4 = .00104976$. When a four-place table is used, such inaccuracies are quite likely to occur. They can be avoided only by the use of a more extended table.

Raise to the indicated power:

3. 8^3.
4. 15^4.
5. 78^2.
6. 3.5^2.
7. 1.07^3.
8. $.028^2$.
9. 2.06^4.
10. $.321^3$.
11. 574^5.
12. $.7854^2$.
13. $.0392^3$.
14. $\left(\frac{276}{1728}\right)^4$.

482. The logarithm of the root of a number is equal to the logarithm of the number divided by the index of the root.

Thus,
$$\sqrt{10000} = \sqrt{10^4} = 10^{\frac{4}{2}}.$$
$$\therefore \log \sqrt{10000} = 4 \div 2 = 2.$$

In general, if $\log m = x$ and $a =$ any root index, we have

$$\sqrt[a]{m} = \sqrt[a]{10^x} = 10^{\frac{x}{a}}. \qquad \text{(Art. 292, III)}$$
$$\therefore \log \sqrt[a]{m} = x \div a = \frac{\log m}{a}.$$

1. Find the cube root of 5.832, and also of 0.05832.

(*a*)

$$\log 5.832 = 0.7658.$$
$$\log 5.832 \div 3 = 0.2553.$$
$$\text{Corresponding number} = 1.8 = \sqrt[3]{5.832}.$$

(*b*)

$$\log 0.05832 = 0.7658 - 2.$$

In this case the characteristic (-2) is not divisible by 3, but by adding and subtracting 1, we may write $0.7658 - 2$ in the form $1.7658 - 3$.

Then $\log 0.05832 \div 3 = .5886 - 1$, or $\bar{1}.5886$.

Corresponding number = .3878.

In a similar manner we can always make a negative characteristic a multiple of the divisor.

Find the root indicated:

2. $289^{\frac{1}{2}}$.
3. $512^{\frac{1}{3}}$.
4. $784^{\frac{1}{2}}$.
5. $729^{\frac{1}{3}}$.
6. $.4096^{\frac{1}{4}}$.
7. $.0635^{\frac{1}{3}}$.
8. $1.728^{\frac{1}{3}}$.
9. $4096^{\frac{1}{3}}$.
10. $\sqrt{2}$.
11. $\sqrt[3]{2}$.
12. $\sqrt{3}$.
13. $\sqrt[5]{3}$.
14. $\sqrt[4]{9}$.
15. $\sqrt{.8}$.
16. $\sqrt[3]{.064}$.
17. $\sqrt{20\frac{1}{4}}$.

Find the value of:

18. $\dfrac{128}{32 \times .7854}$.
19. $\dfrac{75^2}{42 \times 36 \times 3.1}$.
20. $\dfrac{42 \times \sqrt{(.0016)^7}}{\sqrt[8]{108}}$.
21. $\sqrt[3]{\left(\dfrac{294 \times 125}{42 \times 32}\right)^2}$.
22. $\sqrt{\dfrac{.324 \times 68^3}{140 \times 320}}$.
23. $\sqrt[3]{\dfrac{76.4 \times 560 \times 4.02}{3.1416 \times 12.5}}$.

SOME APPLICATIONS OF LOGARITHMS

483. An equation in which the unknown number occurs as an exponent is called an **Exponential Equation.**

Thus, $11^x = 3$ is an exponential equation.

Equations of this kind are solved by taking the logarithms of both members, and then proceeding by the ordinary methods.

1. Find the value of x in the equation $11^x = 3$.

$$11^x = 3.$$

$$\therefore \log 11^x = \log 3.$$

By Art. 481, $$x \log 11 = \log 3.$$

$$\therefore x = \frac{\log 3}{\log 11} = \frac{.4771}{1.0414} = 0.458+.$$

2. Given a, r, l, in a geometrical progression, find n. Calculate the value of n, when $a = 1$, $r = 3$, and $l = 729$.

$$l = ar^{n-1}. \qquad \text{(Art. 429)}$$

$$\therefore \log l = \log a + (n-1)\log r. \qquad \text{(Arts. 478, 481)}$$

$$\therefore n = \frac{\log l - \log a}{\log r} + 1.$$

$$\log 729 = 2.8627\,;\ \log 1 = 0\,;\ \log 3 = 0.4771.$$

$$\therefore n = \frac{2.8627 - 0}{0.4771} + 1 = 6 + 1 = 7.$$

Solve the following equations:

3. $6^x = 36$.
4. $.8^x = .64$.
5. $4^x = 20$.
6. $.3^x = .027$.
7. $7^x = 35$.
8. $12^x = .372$.
9. $8.1^x = 10$.
10. $(0.5)^{-x} = 4$.
11. $(1.6)^{-x} = 8$.

12. Given a, r, s, in a geometrical progression, show that

$$n = \frac{\log [a + (r-1)s] - \log a}{\log r},$$

and find the value of n when $a = 1$, $r = 2$, $s = 511$.

13. Given r, l, s, show that

$$n = \frac{\log l - \log [lr - (r-1)s]}{\log r} + 1,$$

and find the value of n when $r = 5$, $l = 1250$, $s = 1562$.

14. Given a, l, s, show that

$$n = \frac{\log l - \log a}{\log (s-a) - \log (s-l)} + 1,$$

and find the value of n when $a = 343$, $l = \frac{1}{7}$, $s = 400\frac{1}{7}$.

494. Compound Interest is conveniently computed by the use of logarithms. If we represent the *amount* by A, the *principal* by P, and the *rate per cent* by r, we have

in 1 year, $A = P(1 + r)$,
in 2 years, $A = P(1 + r)^2$,
in 3 years, $A = P(1 + r)^3$,
in n years, $\boldsymbol{A = P(1 + r)^n}$. (1)

If the interest is compounded semiannually, as is customary in savings banks, $A = P(1 + \frac{1}{2}r)^{2n}$. (2)

If the interest is compounded quarterly,

$$A = P(1 + \tfrac{1}{4}r)^{4n}. \quad (3)$$

If we take logarithms, formula (1) becomes

$$\log A = \log P + n \log (1 + r). \quad (4)$$

1. Find the amount of \$875 for 12 years at 6% compound interest.

Here $P = 875,\ n = 12,\ r = 6.$

$\therefore \log A = \log 875 + 12 \log 1.06$

$= 2.9420 + 12 \times 0.0253$

$= 3.2456.$

$\therefore A = \$1760.40.$ (28¢ less than by comp. int. table.)

Find the amount at compound interest:

2. Of \$425 for 8 years at 6%.

3. Of \$630 for 10 years at 4%.

4. Of \$785 for 15 years at 5%.

5. Of \$1000 for 20 years at 3½%.

6. Of \$5000 for 9 years 6 months, interest being compounded semiannually.

7. What principal will amount to \$252.82 in 19 years at 4% compound interest?

From formula (4) we find

$$\log P = \log A - n \log (1 + r).$$

Find the principal that will

8. Amount to \$1000 in 6 years at 4% compound interest.

9. Amount to \$18,500 in 11 years at 5% compound interest.

10. Amount to \$9275 in 15 years at 6% compound interest.

11. What is the rate per cent when \$ 500 bearing compound interest for 5 years amounts to \$ 669.12?

From formula (4) we find

$$\log (1 + r) = \frac{\log A - \log P}{n}$$

12. Find the rate of compound interest if \$ 4625 amounts to \$ 7404.80 in 12 years.

13. At 4%, in what time will \$ 1800 amount to \$ 3646.48 at compound interest?

From formula (4) we have $n = \dfrac{\log A - \log P}{\log (1 + r)}$.

14. In what time will \$ 1280 amount to \$ 2500 at 5% compound interest?

15. If \$ 300 is invested at 4% compound interest at a child's birth, what will it be worth when the child is 21 years old?

485. An **Annuity** is a sum of money payable at stated intervals, annually, semiannually, quarterly, etc. Deferred payments bear interest.

486. To find the amount of an unpaid annuity at compound interest.

Suppose an annuity of a dollars is payable Jan. 1 of each year, and that r per cent compound interest is allowed on all deferred payments. If no payments are made, the *amount due* at various times is seen in the following:

Jan. 1, a dollars.

Dec. 31, a dollars and int. for 1 year $= a(1 + r)$.

2d, Jan. 1, $a + a(1 + r)$.

2d, Dec. 31, $a + a(1 + r)$ and int., or $a(1 + r) + a(1 + r)^2$.

3d, Jan. 1, $a + a(1 + r) + a(1 + r)^2$.

3d, Dec. 31, $a(1 + r) + a(1 + r)^2 + a(1 + r)^3$.

nth, Jan. 1, $a + a(1+r) + a(1+r)^2 + a(1+r)^3 + \cdots a(1+r)^{n-1}$, or

$$a[1 + (1+r) + (1+r)^2 + (1+r)^3 + \cdots + (1+r)^{n-1}].$$

This is a geometrical progression whose ratio is $1 + r$. Taking its sum (Art. 430), and letting A represent the amount due at the time of the nth payment, we have

$$A = \frac{a}{r}[(1 + r)^n - 1].$$

1. Find the amount of an annuity of \$ 250 remaining unpaid for 8 years at 4% compound interest.

Here $(1 + r)^n = 1.04^8$.

$\log 1.04^8 = 8 \log 1.04 = .1360.$

Corresponding number $= 1.368+.$

$\therefore (1 + r)^n - 1 = 1.368 - 1 = .368.$

$\log A = \log a - \log r + \log [(1 + r)^n - 1]$

$= \log 250 - \log .04 + \log .368.$

$\log 250 = 2.3979$

$\operatorname{colog} .04 = 11.3979 - 10$

$\log .368 = 0.5658 - 1$

$\therefore \log A = 14.3616 - 11 = 3.3616.$

Hence A, the required amount, is \$2300 (by four-place logs).

2. Find the amount of an annuity of \$ 500 remaining unpaid for 10 years at 5% compound interest.

487. Interest-bearing annuities are sometimes not payable until after the expiration of a certain period. The *present value* of such annuity is the sum which if put at interest for the given time would amount to the value of the annuity. It is represented by P in Art. 484 (1).

488. To find the present value of an annuity.

From Art. 484, we have $A = P(1 + r)^n.$ (1)

From Art. 486, we have $A = \frac{a}{r}[(1 + r)^n - 1].$ (2)

Equating (1) and (2), we have

$$P(1 + r)^n = \frac{a}{r}[(1 + r)^n - 1].$$

$$\therefore P = \frac{a}{r}[(1 + r)^n - 1] \div (1 + r)^n = \frac{a}{r}\left[1 - \frac{1}{(1 + r)^n}\right].$$

1. Find the present value of an annuity of \$300 to run 10 years at 6% compound interest.

Here $(1 + r)^n = 1.06^{10}$.

$\log 1.06^{10} = 10 \log 1.06 = .2530.$

antilog $= 1.79+.$

$\therefore (1 + r)^n - 1 = 1.79 - 1 = .79.$

$\log P = \log a - \log r + \log [(1 + r)^n - 1] - \log (1 + r)^n$

$= \log 300 - \log .06 + \log .79 - \log 1.79.$

$\log 300 = 2.4771$

$\operatorname{colog} .06 = 11.2218 - 10$

$\log .79 = 0.8976 - 1$

$\operatorname{colog} 1.79 = 9.7471 - 10$

$\therefore \log P = 24.3436 - 21 = 3.3436.$

antilog $= 2206.$

Hence $P = \$2206.$

The calculation of $(1 + r)^n$ for r small and n large (by means of four-place tables) is not very accurate for the last figure.

2. What is the present value of an annuity of \$400 for 6 years at 4% compound interest?

3. To what sum will an annuity of \$100 amount in 20 years at 5% compound interest?

4. What is the present value of an annuity of \$10 payable in 20 years, allowing compound interest at 6%?

5. On my boy's 10th birthday, and annually thereafter until he is 21 years of age, I deposit \$5 for him in a savings bank that allows 4% compound interest. What sum has he to his credit on his 21st birthday?

THE BINOMIAL THEOREM

PROOF FOR POSITIVE INTEGRAL EXPONENTS

489. The Binomial Theorem has already been found inductively, and the laws relating to the exponents and coefficients of the letters in the formula by which it is expressed have been stated (Arts. 119, 264). A proof of the theorem for positive integral exponents will now be given.

490. THEOREM. **If the binomial $a + b$ is raised to the nth power, n being a positive integer, the result is expressed by the formula**

$$(a+b)^n \equiv a^n + na^{n-1}b + \frac{n(n-1)}{2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{2\cdot 3}a^{n-3}b^3 + \cdots.$$

I. The theorem is evidently true for $n = 1$, since

$$(a+b)^1 \equiv a+b. \qquad \text{(Art. 288)} \quad (1)$$

It is also true for the 2d power, since by multiplication

$$(a+b)^2 \equiv a^2 + 2ab + b^2 \qquad (2)$$
$$\equiv a^2b^0 + 2a^1b^1 + a^0b^2.$$

By multiplication we also know that

$$(a+b)^3 \equiv a^3 + 3a^2b + 3ab^2 + b^3 \qquad (3)$$
$$\equiv a^3b^0 + 3a^2b^1 + 3a^1b^2 + a^0b^3.$$

II. Assume, for the present, that the theorem is true when n has the particular value r; that is, for the rth power. We shall then have

$$(a+b)^r \equiv a^r + ra^{r-1}b + \frac{r(r-1)}{2}a^{r-2}b^2 + \frac{r(r-1)(r-2)}{1\cdot 2\cdot 3}a^{r-3}b^3 + \cdots. \quad (4)$$

Multiplying both members of (4) by $a+b$, we have

$$\begin{aligned}(a+b)^{r+1} &\equiv a^{r+1} + ra^r b + \frac{r(r-1)}{2}a^{r-1}b^2 \\ &\quad + 1\,a^r b + \quad ra^{r-1}b^2 \\ &\quad + \frac{r(r-1)(r-2)}{2\cdot 3}a^{r-2}b^3 + \cdots \\ &\quad + \quad \frac{r(r-1)}{2}a^{r-2}b^3 \\ &\equiv a^{r+1} + (r+1)a^r b + \frac{r(r+1)}{2}a^{r-1}b^2 \\ &\quad + \frac{(r+1)r(r-1)}{2\cdot 3}a^{r-2}b^3 + \cdots. \quad (5)\end{aligned}$$

Upon comparison this result is seen to be of the same form as the rth power, for replacing r by $r+1$ in the second member of (4), we get

$$a^{r+1} + (r+1)a^r b + \frac{(r+1)(r+1-1)}{1\cdot 2}a^{r-1}b^2 + \frac{(r+1)(r+1-1)(r+1-2)}{1\cdot 2\cdot 3}a^{r-2}b^3 + \cdots,$$

or after simplifying,

$$a^{r+1} + (r+1)a^r b + \frac{(r+1)r}{2}a^{r-1}b^2 + \frac{(r+1)r(r-1)}{2\cdot 3}a^{r-2}b^3 + \cdots,$$

which is identical with the second member of (5), and hence is identical with $(a+b)^{r+1}$.

Hence it is seen that if the theorem were true for the rth power, it would be true for the $(r+1)$th power.

III. But the theorem *has been shown to be true* for the 1st, 2d, and 3d powers, by step I. Therefore, by II, it must be true for the $(3+1)$th, or 4th power, and therefore for the $(4+1)$th, or 5th power; and so on for all positive integral powers.

It can be proved that the theorem is true when n is negative or fractional, but the proof is too difficult for this stage.

491. The method of proof employed in the last article is called **Mathematical Induction.** It is to be noticed that there are two distinct parts of the proof as applied above. In the one step, we are to show that, if the theorem is true for any particular value of n, as $n = r$, then it will also be true for the next value $n = r+1$. In the other step, it is necessary to show that the theorem is true for some initial value of n, as for $n = 1$, $n = 2$, or $n = 3$, so that there may be a starting-point free from "ifs" from which to pass by the induction process to the next higher power, and from this to the next higher power, and so on.

492. The expansion of $(a-b)^n$ can be written from that of $(a+b)^n$ by noting that

$$(a-b)^n \equiv [a+(-b)]^n$$
$$\equiv a^n + na^{n-1}(-b) + \frac{n(n-1)}{1\cdot 2}a^{n-2}(-b)^2 + \cdots$$
$$\equiv a^n - na^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 - \cdots.$$

Observe that when b is negative the terms containing *even* powers of b are positive, and the terms containing *odd* powers are negative.

493. Useful application of the Binomial Theorem has already been made in raising binomials to various powers, but a few exercises are here given for additional practice. By assuming that the theorem is true, whether n is positive, negative, integral, or fractional, the exercises are extended to include the expansion of negative and fractional exponents.

1. Expand $(1+x)^{\frac{1}{2}}$ to 3 terms.

Comparing $(1+x)^{\frac{1}{2}}$ with the formula, we see that $a=1$, $b=x$, and $n=\frac{1}{2}$.

$$\therefore (1+x)^{\frac{1}{2}} = 1^{\frac{1}{2}} + \tfrac{1}{2}\times 1^{-\frac{1}{2}}x + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}\times 1^{-\frac{3}{2}}x^2 + \cdots$$
$$= 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \cdots.$$

Does the formula in this case enable us to find the square root of the given expression to any required number of terms?

2. Expand $(1+x)^{-3}$ to 3 terms.

Here $a=1$, $b=x$, and $n=-3$.

$$\therefore (1+x)^{-3} = 1 + (-3)x + \frac{(-3)(-4)}{1\cdot 2}x^2 + \frac{(-3)(-4)(-5)}{1\cdot 2\cdot 3}x^3 + \cdots$$
$$= 1 - 3x + 6x^2 - 10x^3 + \cdots.$$

3. Expand $(x^{-\frac{3}{5}} - 3y^{\frac{2}{3}})^{-\frac{5}{3}}$ to 3 terms.

Here $a = x^{-\frac{3}{5}}$, $b = -3y^{\frac{2}{3}}$, and $n = -\frac{5}{3}$.

$$\therefore (x^{-\frac{3}{5}} - 3y^{\frac{2}{3}})^{-\frac{5}{3}} = (x^{-\frac{3}{5}})^{-\frac{5}{3}} + (-\tfrac{5}{3})(x^{-\frac{3}{5}})^{-\frac{5}{3}-1}(-3y^{\frac{2}{3}})$$
$$+ \frac{(-\frac{5}{3})(-\frac{5}{3}-1)}{1\cdot 2}(x^{-\frac{3}{5}})^{-\frac{5}{3}-2}(-3y^{\frac{2}{3}})^2 + \cdots$$
$$= x + 5x^{\frac{3}{5}}y^{\frac{2}{3}} + 20x^{\frac{11}{5}}y^{\frac{4}{3}} + \cdots.$$

4. Extract the square root of 3.

$$\sqrt{3} = (4-1)^{\frac{1}{2}}$$
$$= 4^{\frac{1}{2}} + \tfrac{1}{2}\times 4^{-\frac{1}{2}}(-1) - \tfrac{1}{8}\times 4^{-\frac{3}{2}}(-1)^2 + \tfrac{1}{16}\times 4^{-\frac{5}{2}}(-1)^3 - \cdots$$
$$= 2 - \tfrac{1}{4} - \tfrac{1}{64} - \tfrac{5}{512} - \cdots$$
$$= 2 - .25 - .0156^{+} - .0019^{+} - \cdots$$
$$= 2 - .2675^{+}$$
$$= 1.7324^{+}.$$

This is correct to three decimal places, and by subtracting more terms from 2, the root can be found correct to any required number of places.

Expand the following:

5. $(1+2x)^5$.

6. $(a+3b)^6$.

7. $(x-3)^7$.

8. $(a^2+b)^6$.

9. $(2x-3y)^4$.

10. $(m^3+n^2)^5$.

To expand a trinomial by the Binomial Theorem we put two of the terms equal to a single term, then expand, and finally replace the term substituted by the two terms. Thus, $(1 + x + x^2)^4$ may be written $(m + x^2)^4$, and after expansion m may be replaced by $1 + x$.

Expand the following:

11. $(1+x+x^2)^4$.
12. $(a+b+y)^3$.
13. $(x+y+a^2)^5$.
14. $(1+x-y^2)^4$.

Expand to three terms:

15. $(1+x)^{-2}$.
16. $(1+x)^{-\frac{1}{2}}$.
17. $(1-x)^{-3}$.
18. $(1-x)^{-\frac{1}{2}}$.
19. $(3a-x^2)^{-3}$.
20. $(3a-2x)^{\frac{2}{3}}$.
21. $(x^2-2y)^{\frac{2}{3}}$.
22. $(1-3x)^{-\frac{2}{3}}$.
23. $(x^{-2}+y^{-3})^3$.
24. $\sqrt{1+2x}$.

25. Expand to 5 terms $(1-x)^{-4}$.
26. Expand to 5 terms $\left(2x+\frac{b}{3}\right)^7$.
27. Expand to 4 terms $(3m-2n)^6$.
28. Expand to 6 terms $(a-1)^{12}$.
29. Expand to 5 terms $(1-x^2)^{10}$.
30. Expand to 8 terms $(a+2\sqrt{x})^{15}$.
31. Expand to 7 terms $(x^2-1)^{20}$.
32. Expand to 5 terms $(2a+3\sqrt{b})^{-4}$.
33. Expand to 4 terms $(\sqrt{x}-\sqrt{y})^8$.

Extract the square root of the following by expanding a binomial to four terms, reducing the terms to decimals and adding:

34. $\sqrt{2}$.
35. $\sqrt{5}$.
36. $\sqrt{10}$.
37. $\sqrt{8}$.
38. $\sqrt{12}$.
39. $\sqrt{20}$.
40. $\sqrt{99}$.
41. $\sqrt{101}$.

SUPPLEMENTARY PROBLEMS

494. 1. Factor the expression $x^3 - x^2 - x + 1$.

2. Expand to 4 terms $(a + x)^{-\frac{1}{4}}$.

3. Multiply $a^m + a^{\frac{m}{2}} + 1$ by $a^{-m} + a^{-\frac{m}{2}} + 1$.

4. Solve the equation $\frac{x}{a} + \frac{x}{b-a} = \frac{a}{b+a}$.

5. Solve the system $y - x = 4$, $\frac{7-2x}{5-3y} = \frac{3}{2}$.

6. Rationalize the denominator of $\frac{1}{1+\sqrt{-2}+\sqrt{-3}}$.

7. Simplify $\frac{5}{1+2x} - \frac{3x}{1-2x} + \frac{4-13x}{4x^2-1}$.

8. Find the H. C. F. of $2x^4 - 3x^3 + 2x^2 - 2x - 3$ and $4x^4 + 3x^2 + 4x - 3$.

9. Divide the number n into three parts such that the first shall be to the second as $a : b$, and the second to the third as $c : d$.

10. A laborer dug 2 trenches, one of which was 6 meters longer than the other, for 356 dimes, and the digging of each of them cost as many dimes a meter as there were meters in its length. Find the length of each trench.

---

11. Simplify

$$a^3 - \{2a^3 + 3a - [a^2 + 2a + (a-4) - 2a^3 - (a^2 - a)] + 17a^2\}.$$

12. Factor the expression $x^3 - 2x^2 - 3x + 6$.

13. Solve the equation $x^{-2} - 2x^{-1} - 99 = 0$.

14. Divide $x + x^{\frac{2}{3}}y^{\frac{1}{3}} - 2y$ by $x^{\frac{2}{3}} + 2x^{\frac{1}{3}}y^{\frac{1}{3}} + 2y^{\frac{2}{3}}$.

15. Find the value (to 4 decimal places) of

$$(2+\sqrt{2}) \times \frac{2\sqrt{2}-1}{4\sqrt{2}+1}.$$

16. Solve the system $-ax+y=a$, $-by+z=b$, $-cz+x=c$.

17. Rationalize the denominator of $\dfrac{2a^2}{\sqrt{a^2+b^2}+\sqrt{a^2-b^2}}$.

18. A basket of oranges was bought at the rate of 15 cents a dozen. If 5 dozen were given away and the rest were sold at 2 for 5 cents, and this gave a profit of 30 cents on the basket, how many oranges did the basket contain?

19. The figure 7 is exactly midway between the hour hand and the minute hand, and the time is between 6 and 7 o'clock. Required the time.

20. Two persons, A and B, could finish a piece of work in a days. They work together b days, when A is called away, and B finishes in c days. In what time could each do it alone? In what time if $a=3$, $b=1$, $c=4$?

---

21. Multiply $x+3-2\sqrt{-1}$ by $x+3+2\sqrt{-1}$.

22. Show that squaring both sides of $\sqrt{x+5}=8$ is equivalent to multiplying both members of $\sqrt{x+5}-8=0$ by a certain factor.

23. Solve the system $\dfrac{x}{a}+\dfrac{y}{b}=1$, $\dfrac{a}{x}+\dfrac{b}{y}=4$.

24. Reduce the fraction $\dfrac{5x^2+4x-1}{20x^2+21x-5}$ to lowest denominator.

25. Simplify $\left\{\dfrac{x+2y}{x+y}+\dfrac{x}{y}\right\} \div \left\{\dfrac{x+2y}{y}-\dfrac{x}{x+y}\right\}$.

26. Find the H. C. F. of $6x^4-13x^3+19x^2-7x-5$ and $10x^5-19x^4+41x^3-33x^2+36x-35$.

27. Extract the square root of $14+6\sqrt{5}$.

28. Divide the fraction $\frac{a}{b}$ into two parts, such that the numerators of the parts taken together shall be equal to their denominators taken together. What are the parts if the given fraction is $\frac{8}{5}$?

29. When two motor cars meet, they pass each other at 75 miles an hour. One has 90 miles to go, and increases its speed by 5 miles an hour; the other, which has only half as far to go, keeps its original rate and arrives at its destination 18 minutes before the other. What was the original rate of each?

30. A and B run 2 kilometers. First, A gives B a start of 50 meters and beats him by 30 seconds, but in the second heat A gives B a start of 1 minute and is beaten by 60 meters. Find the time in which A and B can each run a kilometer.

---

31. Simplify $\frac{a+x}{(m+n)^3} \times \frac{x^2-y^2}{12} \div \left\{ \frac{m-n}{(m+n)^2} \times \frac{x+y}{6\,(m^2-n^2)} \right\}$.

32. Solve the equation $\frac{x+8}{x+5} - \frac{x+10}{x+7} = \frac{x+7}{x+4} - \frac{x+9}{x+6}$.

33. Solve the system $x+y=3,\ x^4+y^4=17$.

34. Solve the system $\frac{xy}{x+y}=a,\ \frac{xz}{x+z}=b,\ \frac{yz}{y+z}=c$.

35. Insert three geometric means between $\frac{1}{2}$ and 128.

36. Determine all the roots of the equation

$$(x-1)(x+2)(x^2-6x+9)=0.$$

37. Solve the equation $\frac{1}{x+\sqrt{2-x^2}} + \frac{1}{x-\sqrt{2-x^2}} = \frac{x}{2}$.

38. Prove that

$$(x-y)^3+(y-z)^3+(z-x)^3 \equiv 3\,(x-y)(y-z)(z-x).$$

39. Find three numbers such that if the first be multiplied by the sum of the second and third, the second by the sum of the first and third, and the third by the sum of the first and second, the products shall be respectively a, b, and c. Find the numbers, if $a=26,\ b=50,\ c=56$.

40. The length of a field is twice its width; another field which is 50 meters longer and 10 meters broader contains 6800 square meters more than the former. Find the dimensions of each field.

---

41. A dealer put 6 dozen oranges into a certain number of boxes. If he had used 2 boxes more, each would have contained 3 oranges less. How many boxes were used?

42. Show that $(-1-\sqrt{-3})^3 = 8$.

43. Solve the equation $x + \frac{1}{x} = a + \frac{1}{a}$.

44. Solve the equation $4\sqrt[4]{x} + \sqrt{x} = 21$.

45. Solve the system $ax^2 + by^2 = c$, $\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{1}{c}$.

46. Find the L. C. M. of $20x^4 + x^2 - 1$ and $25x^4 + 5x^3 - x - 1$.

47. Multiply $\frac{a}{2}\sqrt{2} - \frac{b}{3}\sqrt{3}$ by $\frac{a}{2}\sqrt{2} + \frac{b}{2}\sqrt{3}$.

48. Factor (1) $3x^2 - 31x + 56$; (2) $p^3q^3 + (p^2 + q^2)^3$.

49. The respective ages of A and B are now in the ratio of 6 : 5, but 22 years hence their ages will be in the ratio of 8 : 7. What is the age of each now?

50. How many liters of water must be mixed with 80 liters of spirits which cost 15 shillings a liter, so that by selling the mixture at 12 shillings a liter there may be a gain of 10% on the outlay?

---

51. Simplify $\frac{a\sqrt{5a^3}}{2\sqrt{5^{-3}a}}$.

52. Find the L. C. M. of $x^3 - 8x + 3$ and $x^6 + 3x^5 + x + 3$.

53. From $(a-x)\sqrt{a^2 - x^2}$ subtract $(a-x)\sqrt{\frac{a+x}{a-x}}$.

54. Solve the system $x(y+1) = 8$, $y(x+2) = 12$.

55. Simplify $2\sqrt[3]{\frac{1}{4}} + 3\sqrt[3]{54}$.

56. Simplify $\dfrac{a^2-(b-c)^2}{(b+c)^2-a^2}\times\dfrac{b^2-(c-a)^2}{(c+a)^2-b^2}\times\dfrac{c^2-(a-b)^2}{(a+b)^2-c^2}$.

57. Solve the equation $32^x=8$.

58. At what time between 1 and 2 o'clock is the minute hand 1 minute-space beyond the hour hand?

59. What two numbers whose difference is d are to each other as $a:b$?

60. A and B start at the same time from two towns, M and N, respectively, and travel towards each other. When they meet it is found that B has traveled a miles more than A, and that it will take A b days longer to reach the town from which B started, and B c days longer to reach the town from which A started. Find the distance between M and N.

---

61. Factor (1) $a^{12}-b^{12}$; (2) $x^4+121y^4-38x^2y^2$.

62. Find the value of $\dfrac{(15.6)^{\frac{1}{2}}\times(.0045)^{\frac{3}{4}}}{(.00065)^{\frac{1}{2}}}$.

63. Solve the equation $\sqrt{5x-1}-\sqrt{8-2x}=\sqrt{x-1}$.

64. Solve the system

$$\frac{8}{\sqrt{x-3}}-\frac{3}{\sqrt{y+3}}=1,\quad \frac{4}{\sqrt{x-3}}+\frac{9}{\sqrt{y+3}}=4.$$

65. Form the quadratic equations whose roots are

(1) $3, -\frac{2}{7}$; (2) $-6, -4$; (3) $a+b, a-b$.

66. Expand to 4 terms $\left(1+\dfrac{a}{x^2}\right)^n$.

67. Simplify (1) $\sqrt{20}+\sqrt{45}+\sqrt{\frac{4}{5}}$;

(2) $\dfrac{\sqrt{1+a}}{1+\sqrt{1+a}}+\dfrac{\sqrt{1-a}}{1-\sqrt{1-a}}$.

68. The product of two numbers, their sum, and the difference of their squares are equal to the same number. Find the numbers.

69. A watch gains as much as a clock loses, and 1799 hours by the clock are equivalent to 1801 hours by the watch. Find how much the watch gains in an hour.

70. A and B go to market, A with melons and B with three times as many apples. They find that if B gives all his apples for the melons, A will lose 10 cents, according to the prevailing prices; therefore A gives only $\frac{2}{5}$ of his melons for the apples, and B loses 6 cents, according to the same prices. But B, selling the melons received from A at 6 cents each, gains by the entire transactions the price of 6 apples. Required the number of apples, of melons, and the prevailing price of each.

---

71. Simplify $\dfrac{(a+d)bc}{(a-b)(a-c)}+\dfrac{(b+d)ca}{(b-c)(b-a)}+\dfrac{(c+d)ab}{(c-a)(c-b)}$.

72. Solve the equation $\dfrac{a}{x-a}-\dfrac{b}{x-b}=\dfrac{a-b}{x-c}$.

73. Solve the system $x^2-xy+y^2=7,\ x^4+x^2y^2+y^4=133$.

74. Solve the system $\dfrac{1}{x}+\dfrac{1}{y}=a,\ \dfrac{1}{y}+\dfrac{1}{z}=b,\ \dfrac{1}{x}+\dfrac{1}{z}=c$.

75. Solve the equation $\dfrac{ab+x}{b^2}-\dfrac{b^2-x}{a^2b}=\dfrac{x-b}{a^2}-\dfrac{ab-x}{b^2}$.

76. Solve the equation $2(x-1)(x-2)-\sqrt{x^2-3x+6}=20$.

77. Show that $a^2+3b^2>2b(a+b)$.

78. What is the size of a cubical cistern such that if it were made 3 ft. longer, 2 ft. narrower, and of the same depth, it would contain the same amount of water?

79. The sum of three numbers in geometrical progression is 57. If the first is multiplied by $\frac{1}{4}$, the second by $\frac{1}{2}$, and the third by $\frac{5}{9}$, the results form an arithmetical progression. Find the numbers.

80. When the price of maple molasses was three times the price of cane molasses, a merchant made two mixtures of maple molasses and cane molasses, and the prices a gallon were in

the ratio of 9 : 10. He afterwards mixed twice as much maple molasses with the same quantity of cane molasses in each case, and the prices were in the same ratio as before. Required the ratio of the quantities mixed.

---

81. Factor the expression $4(ab+cd)^2-(a^2+b^2-c^2-d^2)^2$.

82. Solve the equation $\sqrt{x}+\sqrt{2x+7}-2\sqrt{x+7}=0$.

83. Solve the equation $3^{\frac{1}{x}}=10$.

84. Solve the system $x+y=2$, $3x+y-z=0$, $x^2+y^2+z^2=10$.

85. Expand the binomial $(2x^{-2}-\frac{1}{8}\sqrt[3]{y})^4$.

86. If $a:b=c:d$, prove that $\dfrac{a^2+c^2}{b^2+d^2}=\dfrac{ac}{bd}$.

87. Extract the square root of $4x^4-12x+4+9x^{-2}-6x^{-3}+x^{-4}$.

88. Draw the graphs of the system $x^2+y^2=25$, $3x+4y=25$, and ascertain that they intersect in the points given by the algebraic solution.

89. At the organization of a certain club the annual dues were a dollars, but the next year the expenses increased by d dollars, while the number of members decreased by n; in consequence the annual dues were increased by b dollars. Required the number of original members.

90. A and B lay out some money in speculation. A disposes of his bargain for \$ 11, and gains as many per cent as B lays out in dollars; B's gain is \$ 36, and it appears that A gains four times as much per cent as B gains. Find the sum invested by each.

---

91. Factor (1) $3x^2-11x+10$; (2) $a^2b^2-4abc-5c^2$.

92. Divide $\left(\dfrac{x^2+y^2}{x^2-y^2}-\dfrac{x^2-y^2}{x^2+y^2}\right)$ by $\left(\dfrac{x-y}{x+y}+\dfrac{x+y}{x-y}\right)$.

93. Solve the system $x^2+xy+x=14$, $y^2+xy+y=28$.

94. Solve the system $x^2 + xy = 15$, $xy - y^2 = 2$.

95. Solve the system $\frac{1}{x} + \frac{1}{y} = a$, $\frac{1}{x^2} + \frac{1}{y^2} = b$.

96. Solve the equation $\frac{3x + \sqrt{4x - x^2}}{3x - \sqrt{4x - x^2}} = 2$.

97. Simplify $\sqrt[3]{2} \times \sqrt[6]{\frac{1}{8}} \times \sqrt[3]{3}$. Also $(a^{\frac{3}{5}}b^{-\frac{2}{5}})^{\frac{2}{3}} \times (a^{-3}b^{-\frac{5}{3}})^{\frac{1}{3}}$.

98. Solve the system $x^3 + y^3 = 13(x + y)$, $x^2 - 2xy + y^2 = x + y - 6$.

99. Find the edge of a cube such that the product of its volume increased by unity, the area of one of its faces increased by unity, and one of its edges increased by unity shall equal 30 times its volume.

100. A man had a cask of wine containing 40 gallons; a dishonest servant drew from the cask a measure of wine and filled the cask with water, repeating this two more times. The theft was then discovered, and upon gauging the cask it was found to contain $16\frac{1}{8}$ gallons of wine and $23\frac{1}{8}$ gallons of water. What was the capacity of the measure used?

---

101. Factor the expression $18x^2 - 27abx - 35a^2b^2$.

102. Free $\frac{(x-y)^{-\frac{1}{4}}}{x^{-4} + y^{-4}}$ of negative exponents.

103. Divide $x^3 - \frac{1}{x^3}$ by $x - \frac{1}{x}$.

104. Square $2\sqrt{-3} - 5\sqrt{-2}$.

105. Solve the equation $x^2 + \left(\frac{a^2}{b} + \frac{b^2}{a}\right)x + ab = 0$.

106. Simplify $\dfrac{x + \dfrac{a}{1 + \dfrac{x}{a}}}{x - \dfrac{a}{1 - \dfrac{x}{a}}}$.

107. Find all the positive integral roots of $13x + 27y = 625$.

108. Solve the system $x^2 + y^2 = b,\ xy = a$.

109. A dealer bought $47\frac{1}{2}$ dozen heads of cabbage at 16 for \$ 1, and the rest at 18 for \$ 1. He sold them all at 15 for \$ 1, and gained \$ 3. How many heads of each sort did he buy?

110. A dealer bought a certain quantity of coal, expecting to sell it at 10% profit. Owing to a decline in the market price he had to sell it for \$ 50 less than he expected, and thus lost 15% on the cost. What did he pay for the coal?

---

111. Divide $3 - \sqrt{-1}$ by $2 + \sqrt{-1}$.

112. Simplify $\sqrt{2} \times \sqrt[3]{3} \times \sqrt[4]{\frac{1}{2}} \times \sqrt[3]{\frac{1}{3}}$.

113. Solve the inequality $5x + 2 > 3x + \frac{1}{2}x - 7$.

114. Solve the equation $\dfrac{\sqrt{a} + \sqrt{a - x}}{\sqrt{a} - \sqrt{a - x}} = \dfrac{1}{a}$.

115. Solve the equation $\sqrt{x + 4} - \sqrt{x - 4} = 4$. Check the result, and explain its significance.

116. Find the limiting value of $\dfrac{(x + 1)(x^3 - 3)}{x^4 - 3x}$ when x increases without limit.

117. A skilled workman saves 8% of his wages. After a strike his wages are raised \$ 1 a week; but the cost of living being increased in the ratio of 10 : 9, he then saves \$ 43.68 a year less than before. What were his weekly wages before the strike?

118. If a hydrogen balloon under a pressure of 20 pounds to the square inch contains 20 liters of hydrogen, what will be the pressure to the square inch if it expands so as to contain 25 liters? (See Art. 416, Ex. 4.)

119. If a liter of air under a pressure of 1033 grams to 1 square centimeter weighs 0.001293 gram, what will be the weight of a liter of air under a pressure of 2066 grams to 1 square centimeter, the temperature being the same in each case?

UNIVERSITY OF CALIFORNIA LIBRARY

This book is DUE on the last date stamped below.

Fine schedule: 25 cents on third day overdue
50 cents on fourth day overdue
One dollar on seventh day overdue.

OCT 14 1947

MAY 12 1948

8Oct'53CR

SEP 30 1953 LU

28Nov'53BW

NOV 28 1953 LU

1Nov'56JG

REC'D LD

OCT 25 1956

5Feb'57JF

REC'D LD

FEB 2 1957

19NOV'59LR

REC'D LD

NOV 24 1959

11Mar'62WA

REC'D LD

FEB 26 1962

22Jul'65VI

REC'D LD

JUL 22'65 -3 PM

LD 21-100m-12,'33(A2012s16)4120

YB 17261

911976

QA
154
C57

THE UNIVERSITY OF CALIFORNIA LIBRARY